Invertible field transformations with derivatives: necessary and sufficient conditions

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$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

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Introduction

Field transformations are ubiquitous in mathematics & physics !!

- Gauge (global) transformation of fields
- Bogliubov transformation
- Fourier transformation (series) & Laplace transformation
- Galilean, Lorentz, general coordinate transformations
- …

It provides better ways to understand various physical phenomena, and to advance calculations, in particular, to solve more easily differential equations.

Conformal & disformal transformations in gravity

- **Conformal transformation :**

$$\tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu} \quad \xleftrightarrow{\Omega^2 \neq 0} \quad g_{\mu\nu} = \Omega(x)^{-2}\tilde{g}_{\mu\nu}$$

- **Disformal transformation :** **(Bekenstein 1992)**

$$\tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\partial_\mu\phi\partial_\nu\phi, \quad X = -\frac{1}{2}g^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

$$\Updownarrow \quad \det\left(\frac{\partial\tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}}\right) = A\left(A - A_{,X}X + 2B_{,X}X^2\right) \neq 0.$$

(N.B. No derivatives of the metrics)

$$g_{\mu\nu} = \tilde{A}(\phi, \widetilde{X})\tilde{g}_{\mu\nu} + \tilde{B}(\phi, \widetilde{X})\partial_\mu\phi\partial_\nu\phi, \quad \widetilde{X} = -\frac{1}{2}\tilde{g}^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi = \frac{X}{A - 2BX}$$

$$\left(\tilde{A}(\phi, \widetilde{X}) = \frac{1}{A(\phi, X)}, \quad \tilde{B}(\phi, \widetilde{X}) = -\frac{B(\phi, X)}{A(\phi, X)}.\right)$$

Inverse function theorem

Field transformations without derivatives :

$$\tilde{\phi}^a(x^\mu) = \tilde{\phi}^a[\phi^b(x^\mu)]$$

(Local) invertibility $\Longleftrightarrow$ $\det|(\partial\tilde{\phi}^a/\partial\phi^b)| \neq 0$

N.B. this theorem, of course, applies to point particle theories.

$$\tilde{q}^a(t) = \tilde{q}^a[q^b(t)] \Longrightarrow \det|(\partial\tilde{q}^a)/\partial q^b| \neq 0$$

(Invertibility)

How to judge the invertibility for field transformations with derivatives ??

Example

Varaiable transformation from q(t) to Q(t) with derivatives:

e.g. $Q(t) = q(t) + \dot{q}(t)$

Inverse transformation

$$\Rightarrow \quad q(t) = e^{-t}\left(e^{t_0} q(t_0) + \int_{t_0}^{t} e^{t} Q(t)\right).$$

Given Q(t) completely, q(t) is not uniquely determined !!

$\Rightarrow$ **This is not an invertible transformation !!**

(But, we have an example of
an invertible transformation with derivatives)

Example of "invertible" transformation with derivatives

e.g.

$$\begin{cases} \phi_1 &= \psi_1, \\ \phi_2 &= \psi_2 + \eta^{\mu\nu}\partial_\mu\psi_1\partial_\nu\psi_1. \end{cases}$$

$$\Longleftrightarrow \begin{cases} \psi_1 &= \phi_1, \\ \psi_2 &= \phi_2 - \eta^{\mu\nu}\partial_\mu\phi_1\partial_\nu\phi_1. \end{cases}$$

If one constructs the inverse transformation explicitly, it manifestly shows the invertibility.

But, an explicit form of the inverse transformation is not necessarily obtained.

Let's try to extend

the inverse function theorem to

field transformation with derivatives !!

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu\right)$$

But, how ???

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu\right)$$

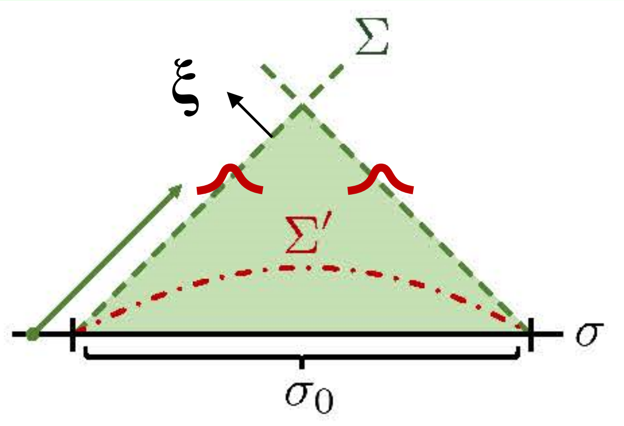
We look on this transformation as differential equations for old variables.

Then, we are going to use the method of characteristics.

Causal structure and invertibility

EOMs give (determine) causal structure through characteristics.

- **D dim spacetime**
  = (D-1) dim surface + the other direction ξ
- **Characteristic surface**
  = on which, the coefficient of the highest-order derivatives of EOMs to ξ-direction becomes zero.



→ **Then, the EOMs cannot be solved uniquely beyond it.**
In fact, this surface coincides with the edge of causality.

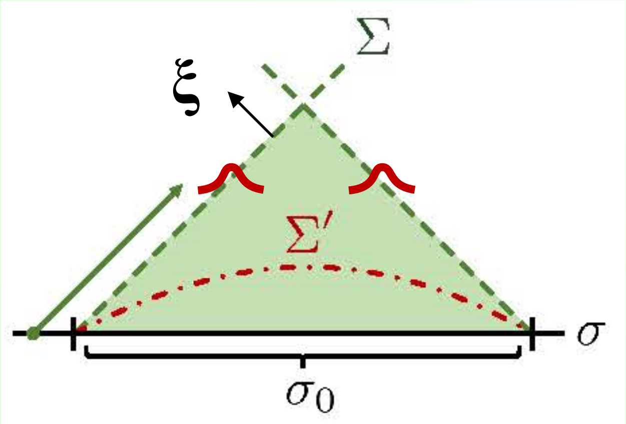
→ **If one-to-one correspondence (*i.e.* invertibility) would be achieved, the two theories must have the same characteristics.**

Method of characteristics (courtesy of Nori)

Characteristic surface Σ = wave propagation surface
(Maximum propagation speed is determined by characteristics)

- **For a quasi-linear (hyperbolic) PDE, pick up the highest derivative terms:**

$$E_\phi = \mathcal{G}^{\mu\nu}(\phi, \partial\phi)\partial_\mu\partial_\nu\phi + \cdots = 0.$$



- **Estimate the principal symbol P(ξ):**

$$P(\xi) \equiv \xi_\mu \xi_\nu \frac{\partial E_\phi}{\partial(\partial_\mu\partial_\nu\phi)} = \xi_\mu\xi_\nu\mathcal{G}^{\mu\nu}$$

$(\mathcal{G}^{\mu\nu} = g^{\mu\nu} \Rightarrow \xi^\mu : \text{null})$

- **Solve the characteristic equation (det)P(ξ) = 0:**

$$P(\xi) = \xi_\mu\xi_\nu\mathcal{G}^{\mu\nu} = 0$$

(2 propagation modes = 1 dof)

Time evolution is uniquely determined in the green region (once an initial data is given on σ_0), but not beyond Σ.

Invertible transformation and characteristics

- **Field transformations from ψ_a to φ_i :**

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu\right). \ (1)$$

We can view Eqs. (1) as the system of differential equations for ψ_a.

e.g.

$$\partial_\sigma\left[\left(\partial_\mu\phi\right)^4\right] = 4\partial^\mu\phi\left(\partial_\nu\phi\right)^2 \partial_\sigma\partial_\mu\phi$$

To make it quasi-linear
by taking additional derivatives:

characteristic matrix:

det P = det M_{11} · det M_{22} = 0

⇔ det M_{11} = 0 or det M_{22} = 0

(Neglect fake characteristics)

(This condition should not generate additional characteristics.)

Identically zero

Let's take a lesson

Lesson

e.g. $$L=-\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1-\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2+\beta\partial_\mu\phi_1\partial^\mu\phi_2$$

$$\Rightarrow\begin{cases}E_{\phi_1}=\Box\phi_1-\beta\Box\phi_2=0\\ E_{\phi_2}=\Box\phi_2-\beta\Box\phi_1=0\end{cases}$$

(4 propagation modes = 2 dof for β≠ 1)

$$P_\phi=\begin{pmatrix}\xi_\mu\xi_\nu\frac{\partial E_{\phi_1}}{\partial(\partial_\mu\partial_\nu\phi_1)} & \xi_\mu\xi_\nu\frac{\partial E_{\phi_1}}{\partial(\partial_\mu\partial_\nu\phi_2)}\\ \xi_\mu\xi_\nu\frac{\partial E_{\phi_2}}{\partial(\partial_\mu\partial_\nu\phi_1)} & \xi_\mu\xi_\nu\frac{\partial E_{\phi_2}}{\partial(\partial_\mu\partial_\nu\phi_2)}\end{pmatrix}=\begin{pmatrix}\xi^2 & -\beta\xi^2\\ -\beta\xi^2 & \xi^2\end{pmatrix}\Rightarrow\det P_\phi=\left(1-\beta^2\right)(\xi^2)^2$$

Invertible transformation : $$\begin{cases}\phi_1=\psi_1+\frac{\alpha}{2}\partial_\mu\psi_2\partial^\mu\psi_2\\ \phi_2=\psi_2\end{cases}$$

$$\Rightarrow\begin{cases}E_{\psi_1}=\Box\psi_1+\alpha\left(\partial_\mu\partial_\nu\psi_2\partial^\mu\partial^\nu\psi_2+\partial_\mu\psi_2\partial^\mu\Box\psi_2\right)-\beta\Box\psi_2=0\\ E_{\psi_2}=\Box\psi_2-\beta\left[\Box\psi_1+\alpha\left(\partial_\mu\partial_\nu\psi_2\partial^\mu\partial^\nu\psi_2+\partial_\mu\psi_2\partial^\mu\Box\psi_2\right)\right]=0\end{cases}$$

$$P_\psi=\begin{pmatrix}\xi_\mu\xi_\nu\frac{\partial E_{\psi_1}}{\partial(\partial_\mu\partial_\nu\psi_1)} & \xi_\mu\xi_\nu\xi_\sigma\frac{\partial E_{\psi_1}}{\partial(\partial_\mu\partial_\nu\partial_\sigma\psi_2)}\\ \xi_\mu\xi_\nu\frac{\partial E_{\psi_2}}{\partial(\partial_\mu\partial_\nu\psi_1)} & \xi_\mu\xi_\nu\xi_\sigma\frac{\partial E_{\psi_2}}{\partial(\partial_\mu\partial_\nu\partial_\sigma\psi_2)}\end{pmatrix}=\begin{pmatrix}\xi^2 & \alpha\xi^2\xi_\mu\partial^\mu\psi_2\\ -\beta\xi^2 & -\alpha\beta\xi^2\xi_\mu\partial^\mu\psi_2\end{pmatrix}\Rightarrow\det P_\psi=0$$

Identically zero

This fact is almost equivalent to det M22 = 0 because the third order derivatives are fakes and do not lead to additional d.o.f. (propagation mode).

Procedures

Field transformations: $\phi_i = \phi_i(\psi_a, \partial_\mu)$ **(i, a = 1, …, n)**

Necessary for invertibility

necessary conditions

Invertibility $\Rightarrow$ **No characteristics for ψa associated with derivatives !!**

- **det P for the highest order derivatives must be identically zero, that is, the characteristic matrix must be degenerate (with m degrees).**
- **After taking adequate linear combinations, m EOMs reduce to lower order differential eqs. We repeat this procedure until the derivatives by transformation vanish.**
- **In the last step, the characteristic matrix should not be degenerate, which corresponds to the condition of inverse function theorem.**

Simplest case

$$\begin{cases} \phi_1 = \phi_1\left(\psi_1, \psi_2, \partial_\mu\psi_1, \partial_\mu\psi_2\right) \\ \phi_2 = \phi_2\left(\psi_1, \psi_2, \partial_\mu\psi_1, \partial_\mu\psi_2\right) \end{cases}$$

Two fields with up to first order derivatives

But, the extension to any number of fields with any order derivatives is straightforward.

Necessary conditions

Necessary conditions for invertibility

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu \psi_a\right) \quad (i, a = 1, 2).$$

Two useful matrices: $A^\mu_{ia} \equiv \dfrac{\partial \phi_i}{\partial(\partial_\mu \psi_a)}, \quad B_{ia} \equiv \dfrac{\partial \phi_i}{\partial \psi_a}.$

- **For all A = 0, the transformation does not involve derivatives.**

  ➔ Invertibility ⇔ det B ≠ 0 (Inverse function theorem).

- **We are interested in the case with at least some A ≠ 0.**

To make it quasi-linear : **e.g.** $\partial_\sigma \left[\left(\partial_\mu \phi\right)^4 \right] = 4 \partial^\mu \phi \left(\partial_\nu \phi\right)^2 \partial_\sigma \partial_\mu \phi$

$$\begin{aligned} C^{(\mu_1 \cdots \mu_n)} \partial_{\mu_1 \cdots \mu_n} \phi_i = & \quad C^{(\mu_1 \cdots \mu_n)} \left[A^{\mu_{n+1}}_{ia} \partial_{\mu_1 \cdots \mu_{n+1}} \psi_a + B_{ia} \partial_{\mu_1 \cdots \mu_n} \psi_a \right] \\ & + \quad n \quad C^{(\mu_1 \cdots \mu_n)} \left[\left(\partial_{\mu_1} A^{\mu_{n+1}}_{ia} \right) \partial_{\mu_2 \cdots \mu_{n+1}} \psi_a + \left(\partial_{\mu_1} B_{ia} \right) \partial_{\mu_2 \cdots \mu_n} \psi_a \right] \\ & + \; {}_n C_2 \; C^{(\mu_1 \cdots \mu_n)} \left[\left(\partial_{\mu_1 \mu_2} A^{\mu_{n+1}}_{ia} \right) \partial_{\mu_3 \cdots \mu_{n+1}} \psi_a + \left(\partial_{\mu_1 \mu_2} B_{ia} \right) \partial_{\mu_3 \cdots \mu_n} \psi_a \right] \\ & + \; \cdots \end{aligned}$$

↑

Arbitrary function

First degeneracy

- **Leading terms (highest derivative terms)** $A_{ia}^{\mu} := \dfrac{\partial \phi_i}{\partial(\partial_\mu \psi_a)}$

$$C^{(\mu_1\cdots\mu_n)}\partial_{\mu_1\cdots\mu_n}\phi_i = C^{(\mu_1\cdots\mu_n)}A_{ia}^{\mu_{n+1}}\partial_{\mu_1\cdots\mu_{n+1}}\psi_a + \cdots \equiv E_{\psi_i}$$

⇒ $$P_\psi = \begin{pmatrix} \xi^{n+1}\dfrac{\partial E_{\psi_1}}{\partial(\partial^{n+1}\psi_1)} & \xi^{n+1}\dfrac{\partial E_{\psi_1}}{\partial(\partial^{n+1}\psi_2)} \\ \xi^{n+1}\dfrac{\partial E_{\psi_2}}{\partial(\partial^{n+1}\psi_1)} & \xi^{n+1}\dfrac{\partial E_{\psi_2}}{\partial(\partial^{n+1}\psi_2)} \end{pmatrix} = \begin{pmatrix} C^{(\mu_1\cdots\mu_n)}A_{11}^{\mu_{n+1}}\xi_{\mu_1}\cdots\xi_{\mu_{n+1}} & C^{(\mu_1\cdots\mu_n)}A_{12}^{\mu_{n+1}}\xi_{\mu_1}\cdots\xi_{\mu_{n+1}} \\ C^{(\mu_1\cdots\mu_n)}A_{21}^{\mu_{n+1}}\xi_{\mu_1}\cdots\xi_{\mu_{n+1}} & C^{(\mu_1\cdots\mu_n)}A_{22}^{\mu_{n+1}}\xi_{\mu_1}\cdots\xi_{\mu_{n+1}} \end{pmatrix}$$

⇒ $$\det P_\psi = \left(C^{(\mu_1\cdots\mu_n)}\xi_{\mu_1}\cdots\xi_{\mu_n}\right)^2 \det\left(A_{ia}^{\mu}\xi_\mu\right) = \mathcal{O}\left(\xi^{2n+2}\right) = 0.$$

(Fake characteristics)

⇒ **2n+2 modes = 2n modes (from $\partial_{\mu_1\ldots\mu_n}$) + 2 modes (from $\partial\psi$)**

⇒ **Invertibility ➔ No characteristics from** $\det\left(A_{ia}^{\mu}\xi_\mu\right)$

⇒ $\forall\xi_\mu,\ \det\left(A_{ia}^{\mu}\xi_\mu\right) = 0$ ⇔ $\det\left(A_{ia}^{(\mu\nu)}\right) \equiv \dfrac{1}{2}\epsilon_{i_1i_2}\epsilon_{a_1a_2}A_{i_1a_1}^{(\mu}A_{i_2a_2}^{\nu)} = 0$

Second degeneracy

- **Sub-leading terms (second highest derivative terms)**

$$
\begin{aligned}
C^{(\mu_1\cdots\mu_n)}\partial_{\mu_1\cdots\mu_n}\phi_i = \; & C^{(\mu_1\cdots\mu_n)}A_{ia}^{\mu_{n+1}}\partial_{\mu_1\cdots\mu_{n+1}}\psi_a \\
+ \; & C^{(\mu_1\cdots\mu_n)}\left[B_{ia}\partial_{\mu_1\cdots\mu_n}\psi_a + n\left(\partial_{\mu_1}A_{ia}^{\mu_{n+1}}\right)\partial_{\mu_2\cdots\mu_{n+1}}\psi_a\right] + \cdots
\end{aligned}
$$

We need to remove the leading term.

- **Define a (transposed) cofactor matrix :** $\overline{A}^{\mu}_{ai} = \begin{pmatrix} A^{\mu}_{22} & -A^{\mu}_{12} \\ -A^{\mu}_{21} & A^{\mu}_{11} \end{pmatrix}$

$$\Rightarrow \quad \overline{A}^{(\mu}_{ai}A^{\nu)}_{ib} = \det\left(A^{(\mu\nu)}_{jc}\right)\cdot\delta_{ab} = 0$$

- **By setting** $C^{(\mu_1\cdots\mu_n)} = \tilde{C}^{(\mu_1\cdots\mu_n)}\overline{A}^{\beta}_{bi}\partial_{\beta},$

symmetric part only

$$
\begin{aligned}
\tilde{C}^{(\mu_1\cdots\mu_n)}\overline{A}^{\beta}_{bi}\partial_{\mu_1\cdots\mu_n\beta}\phi_i = \; & \cancel{\tilde{C}^{(\mu_1\cdots\mu_{n-1})}\overline{A}^{\beta}_{bi}A^{\mu_{n+1}}_{ia}\partial_{\mu_1\cdots\mu_{n+1}\beta}\psi_a} \\
& +\tilde{C}^{(\mu_1\cdots\mu_n)}\overline{A}^{\beta}_{bi}\left[B_{ia}\partial_{\mu_1\cdots\mu_n\beta}\psi_a + \left(\partial_{\beta}A^{\mu_{n+1}}_{ia}\right)\partial_{\mu_1\cdots\mu_{n+1}}\psi_a\right] + \cdots
\end{aligned}
$$

More precisely, …

Sub-leading & subsub-leading

$$
\begin{aligned}
& \tilde{C}^{(\mu_1\cdots\mu_n)}\bar{A}^{\mu_{n+1}}_{ai}\partial_{\mu_1\cdots\mu_{n+1}}\phi_i + n\tilde{C}^{(\beta\mu_1\cdots\mu_{n-1})}\left(\partial_\beta\bar{A}^{\mu_n}_{ai}\right)\partial_{\mu_1\cdots\mu_n}\phi_i \\
& +{}_nC_2\tilde{C}^{(\beta\gamma\mu_1\cdots\mu_{n-2})}\left(\partial_{\beta\gamma}\bar{A}^{\mu_{n-1}}_{ai}\right)\partial_{\mu_1\cdots\mu_{n-1}}\phi_i \\
= \;& \tilde{C}^{(\mu_1\cdots\mu_n)}\mathcal{A}^{\mu_{n+1}}_{2,ab}\partial_{\mu_1\cdots\mu_{n+1}}\psi_b \\
& +\left(\tilde{C}^{(\mu_1\cdots\mu_n)}\mathcal{B}_{2,ab} + n\tilde{C}^{(\beta\mu_1\cdots\mu_{n-1})}\left(\partial_\beta\mathcal{A}^{\mu_n}_{2,ab}\right)\right)\partial_{\mu_1\cdots\mu_n}\psi_b + \mathcal{O}\left(\partial^{n+D-3}\psi\right)
\end{aligned}
$$

$$
\bar{A}^\mu_{ai} = \epsilon_{ab}\epsilon_{ij}A^\mu_{bj}, \quad \mathcal{A}^\mu_{2,ab} \equiv \bar{A}^\mu_{ai}B_{ib} + \bar{A}^\beta_{ai}\left(\partial_\beta A^\mu_{ib}\right), \quad \mathcal{B}_{2,ab} \equiv \bar{A}^\beta_{ai}\left(\partial_\beta B_{ib}\right).
$$

$$
\left(A^\mu_{ia} = \frac{\partial\phi_i}{\partial(\partial_\mu\psi_a)}, \quad B_{ia} = \frac{\partial\phi_i}{\partial\psi_a}\right)
$$

- **No additional characteristics at sub-leading order:**

$$
\mathcal{A}^{(\alpha}_{2,ab}\bar{A}^{\beta)}_{bj} = 0.
$$

- **Inverse function theorem at subsub-leading order:**

$$
\begin{aligned}
\forall\xi_\mu \neq 0, \quad & \Big\{A^{\mu_1}_{kb}A^{\mu_2}_{kb}\mathcal{B}_{2,ac}\bar{A}^{\mu_3}_{cj} - \mathcal{A}^{\mu_1}_{2,ab}A^{\mu_2}_{ib}B_{ic}\bar{A}^{\mu_3}_{cj} - \mathcal{A}^{\mu_1}_{2,ab}\bar{A}^{\alpha}_{bi}\bar{A}^{\mu_2}_{ci}\left(\partial_\alpha\bar{A}^{\mu_3}_{cj}\right) \\
& +\mathcal{A}^{\alpha}_{2,ab}A^{\mu_1}_{ib}A^{\mu_2}_{ic}\left(\partial_\alpha\bar{A}^{\mu_3}_{cj}\right) - \mathcal{A}^{\mu_1}_{2,ab}A^{\mu_2}_{ib}A^{\alpha}_{ic}\left(\partial_\alpha\bar{A}^{\mu_3}_{cj}\right)\Big\}\xi_{\mu_1}\xi_{\mu_2}\xi_{\mu_3} \neq 0
\end{aligned}
$$

Necessary conditions for invertibility

$$\begin{cases} (1)\ \det\left(A_{ia}^{(\mu\nu)}\right) \equiv \frac{1}{2}\epsilon_{i_1 i_2}\epsilon_{a_1 a_2} A_{i_1 a_1}^{(\mu} A_{i_2 a_2}^{\nu)} = 0 \\ (2)\ \mathcal{A}_{2,ab}^{(\alpha}\bar{A}_{bj}^{\beta)} = 0. \\ (3)\ \forall \xi_\mu \neq 0, \quad \left\{ A_{kb}^{\mu_1} A_{kb}^{\mu_2} \mathcal{B}_{2,ac} \bar{A}_{cj}^{\mu_3} - \mathcal{A}_{2,ab}^{\mu_1} A_{ib}^{\mu_2} B_{ic} \bar{A}_{cj}^{\mu_3} - \mathcal{A}_{2,ab}^{\mu_1} \bar{A}_{bi}^{\alpha} \bar{A}_{ci}^{\mu_2} \left(\partial_\alpha \bar{A}_{cj}^{\mu_3}\right) \right. \\ \qquad \left. + \mathcal{A}_{2,ab}^{\alpha} A_{ib}^{\mu_1} A_{ic}^{\mu_2} \left(\partial_\alpha \bar{A}_{cj}^{\mu_3}\right) - \mathcal{A}_{2,ab}^{\mu_1} A_{ib}^{\mu_2} A_{ic}^{\alpha} \left(\partial_\alpha \bar{A}_{cj}^{\mu_3}\right) \right\} \xi_{\mu_1}\xi_{\mu_2}\xi_{\mu_3} \neq 0 \end{cases}$$

can be simplified into

$$\left(A_{ia}^{\mu} = \frac{\partial \phi_i}{\partial(\partial_\mu \psi_a)}, \quad B_{ia} = \frac{\partial \phi_i}{\partial \psi_a} \right)$$

$$\begin{cases} (1)'\ A_{ia}^{\mu} = a^\mu V_i U_a \quad (V_i V_i = 1, \quad U_a U_a = 1) \\ (2)'\ n_i B_{ia} m_a = 0 \\ (3)'\ n_i B_{ia} U_a \neq 0, \quad (V_i B_{ia} - a^\mu \partial_\mu U_a)\, m_a \neq 0 \end{cases}$$

$$\left(n_i := \epsilon_{ij} V_j, \quad m_a := \epsilon_{ab} U_b \right)$$

Sufficient conditions

Perturbative analysis

At some local region, there is a correspondence:

$$\overline{\phi}_i = \overline{\phi}_i\left(\overline{\psi}_a, \partial_\mu \overline{\psi}_a\right) \quad (i, a = 1, 2).$$

Given $\phi_i = \overline{\phi}_i + \delta\phi_i$ **at an adjacent point,**
there might be two correspondences.

$$\begin{cases} \psi_a^{(1)} = \overline{\psi}_a + \delta\psi_a^{(1)} \\ \psi_a^{(2)} = \overline{\psi}_a + \delta\psi_a^{(2)} \end{cases} \Longrightarrow \quad \delta\phi_i = A_{ia}^{\mu}\partial_\mu\delta\psi_a^{(1)} + B_{ia}\delta\psi_a^{(1)} = A_{ia}^{\mu}\partial_\mu\delta\psi_a^{(2)} + B_{ia}\delta\psi_a^{(2)}$$

$$\left(A_{ia}^{\mu} = \frac{\partial\phi_i}{\partial(\partial_\mu\psi_a)}, \quad B_{ia} = \frac{\partial\phi_i}{\partial\psi_a}\right)$$

$$\Psi_a = \delta\psi_a^{(1)} - \delta\psi_a^{(2)} \Longrightarrow 0 = A_{ia}^{\mu}\partial_\mu\Psi_a + B_{ia}\Psi_a$$

We would like to show Ψa = 0 under the necessary conditions.

Perturbative analysis II

$$0 = A^{\mu}_{ia}\partial_{\mu}\Psi_a + B_{ia}\Psi_a$$

Conditions:

$(1)'$ $A^{\mu}_{ia} = a^{\mu}V_iU_a \quad (V_iV_i = 1, \quad U_aU_a = 1)$

$(2)'$ $n_iB_{ia}m_a = 0$

$(3)'$ $n_iB_{ia}U_a \neq 0, \quad (V_iB_{ia} - a^{\mu}\partial_{\mu}U_a)\, m_a \neq 0$

$\left(n_i := \epsilon_{ij}V_j, \quad m_a := \epsilon_{ab}U_b\right)$

$\Rightarrow$ $m_am_b + U_aU_b = \delta_{ab}$

$$0 = a^{\mu}V_iU_a\partial_{\mu}\Psi_a + B_{ia}\Psi_a$$

$\Downarrow \times n_i (= \epsilon_{ij}V_j)$

$$0 = n_iB_{ia}\Psi_a$$

$\Downarrow$

$$\begin{aligned} 0 &= n_iB_{ia}\left(m_am_b + U_aU_b\right)\Psi_b \\ &= \left(n_iB_{ia}U_a\right)\left(U_b\Psi_b\right) \end{aligned}$$

$\Rightarrow$ $0 = U_b\Psi_b$

In the same way,

$$0 = a^{\mu}V_iU_a\partial_{\mu}\Psi_a + B_{ia}\Psi_a$$

$\Downarrow \times V_i$

$$0 = -a^{\mu}\Psi_a\partial_{\mu}U_a + V_iB_{ia}\Psi_a$$

$\Downarrow$

$$0 = \left(V_iB_{ia} - a^{\mu}\partial_{\mu}U_a\right)m_a\left(m_b\Psi_b\right)$$

$\Rightarrow$ $0 = m_b\Psi_b$

$\Rightarrow$ $\left[\begin{array}{l} U_b\partial_{\mu}\Psi_b = -\Psi_b\partial_{\mu}U_b \\ \Psi_a = (m_am_b + U_aU_b)\Psi_b = m_am_b\Psi_b \end{array}\right]$

$$\boxed{\therefore\ \Psi_a = \left(m_am_b + U_aU_b\right)\Psi_b = 0}$$

Concrete (nontrivial) example

Concrete example

(1)' **linear**

$$\phi_i = a^\mu(\psi_b)V_i(\psi_b)U_a(\psi_b)\partial_\mu\psi_a + f_i(\psi_b)$$

$$\xrightarrow[\psi_a=\psi_a(\psi_b^{\text{new}})]{} U_a=(1,0) \Longrightarrow \phi_i = a^\mu(\psi_b)V_i(\psi_b)\partial_\mu\psi_1 + f_i(\psi_b)$$

Conditions:

$$\left\{\begin{array}{l} (1)'\ A^\mu_{ia} = a^\mu V_i U_a \quad (V_iV_i=1,\quad U_aU_a=1) \\ (2)'\ n_iB_{ia}m_a = 0 \\ (3)'\ n_iB_{ia}U_a \neq 0,\quad (V_iB_{ia} - a^\mu\partial_\mu U_a)\,m_a \neq 0 \\ \left(n_i := \epsilon_{ij}V_j,\quad m_a := \epsilon_{ab}U_b\right) \end{array}\right.$$

$$\Longleftarrow$$

$$\left\{\begin{array}{l} (2)'\ n_iB_{ia}m_a = n_i\dfrac{\partial V_i}{\partial\psi_2}a^\mu\partial_\mu\psi_1 + n_i\dfrac{\partial f_i}{\partial\psi_2} = 0 \\ (3)'\ n_iB_{ia}U_a = n_i\dfrac{\partial V_i}{\partial\psi_1}a^\mu\partial_\mu\psi_1 + n_i\dfrac{\partial f_i}{\partial\psi_1} \neq 0 \\ (3)''\ (V_iB_{ia} - a^\mu\partial_\mu U_a)\,m_a = \dfrac{\partial a^\mu}{\partial\psi_2}\partial_\mu\psi_1 + V_i\dfrac{\partial f_i}{\partial\psi_2} \neq 0 \end{array}\right.$$

$$\xrightarrow[{}^\forall\psi_a,\,{}^\forall\partial_\mu\psi_a]{(2)'} n_i\frac{\partial V_i}{\partial\psi_2} = 0,\quad n_i\frac{\partial f_i}{\partial\psi_2} = 0. \Longrightarrow V_i = V_i(\psi_1),\quad {}^\exists g(\psi_1) = n_i(\psi_1)f_i(\psi_a) = n_i\phi_i.$$

$$\xrightarrow{(3)'} h_1(\psi_1,\phi_i) := g(\psi_1) - n_i(\psi_1)\phi_i = 0,\quad \frac{\partial}{\partial\psi_1}h_1(\psi_1,\phi_i) \neq 0. \xrightarrow{\text{Implicit function theorem}} \psi_1 = \psi(\phi_i)$$

$$\xrightarrow{(3)''} h_2(\psi_2,\psi_1(\phi_i),\partial_\mu\psi_1(\phi_i)) := V_i(\psi_1)\phi_i - [a^\mu(\psi_a)\partial_\mu\psi_1 + V_i(\psi_1)f_i(\psi_a)] = 0,\quad \frac{\partial}{\partial\psi_2}h_2(\psi_2,\psi_1(\phi_i),\partial_\mu\psi_1(\phi_i)) \neq 0$$

$$\xrightarrow{\text{Implicit function theorem}} \psi_2 = \psi_2(\phi_i,\psi_1(\phi_i),\partial_\mu\psi_1(\phi_i))$$

Invertible !!

Summary

We have derived the necessary and sufficient conditions for the invertibility of field transformations with derivatives, which is the extension of the inverse function theorem.