From a Massive to a Partially Massless (PM) Graviton on Curved Space-times

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Cédric Deffayet
(IAP and IHÉS, CNRS Paris)

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1. Massive gravity 101:
   Massive graviton on
   Einstein space-times

2. Massive graviton on arbitrary spacetimes.

   L. Bernard, C.D., M. von Strauss + A. Schmidt-May
   (2015-2016, PR**D**, JCAP)

3. PM graviton on non Einstein spacetimes.

   L. Bernard, C.D., K. Hinterbichler and M. von Strauss
   arXiv:1703.02538 (PR**D**)

1. Massive gravity 101: Fierz-Pauli linear theory for an Einstein space-time

Consider an Einstein space-time obeying $R_{\mu\nu} = \Lambda \, g_{\mu\nu}$

Fierz-Pauli theory (1939)

is the (<u>only correct</u>) theory of a massive graviton $h_{\mu\nu}$ which propagates on this space-time

1. Massive gravity 101: Fierz-Pauli linear theory for an Einstein space-time

Consider an Einstein space-time obeying $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$

Fierz-Pauli theory is defined by

Fierz-Pauli (1939), Deser Nepomechie (1984), Higuchi (1987), Bengtsson (1995), Porrati (2001)

Field equations $E_{\mu\nu} \simeq 0$

on shell

with

$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda\left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h\right) + \frac{m^2}{2}\left(h_{\mu\nu} - g_{\mu\nu} h\right)$$

Kinetic operator

Cosmological constant

Mass term

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Comes from expanding the

Einstein-Hilbert action $\int d^4x \sqrt{-g}\,(R - 2\Lambda)$

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Kinetic operator

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Mass term

Comes from expanding the

Einstein-H

Analogous to Proca equations for a massive photon :

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

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Kinetic operator

Cosmological constant

Mass term

Comes from expanding the Einstein-Hilbert action $\int d^4x \sqrt{-g}\,(R - 2\Lambda)$

Interest: Cosmology ?
The mass term leads to a large distance modification of gravity
(C.D., Dvali, Gabadadze 2001)

DOF (= « polarizations ») counting

The Fierz Pauli theory for a massive graviton of mass m propagates

- 2 DOF if $m = 0$

Massless graviton (of GR)

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Generic massive graviton

- 5 DOF if $m \neq 0$ and $m^2 \neq 2\,\Lambda/3$

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- 5 DO

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the Proca « photon »
has 3 polarizations

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- 5 DOF if $m \neq 0$ and $m^2 \neq 2\,\Lambda/3$

- 4 DOF if $m^2 = 2\,\Lambda/3$

Partially Massless graviton

How to count DOF ?

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$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda\left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h\right) + \frac{m^2}{2}\left(h_{\mu\nu} - g_{\mu\nu} h\right)$$

Kinetic operator

Cosmological constant

Mass term

Comes from expanding the

Einstein-Hilbert action $\int d^4x \sqrt{-g}\,(R - 2\Lambda)$

This implies the (Bianchi) offshell identities

$$\nabla^\mu\left[\mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda\left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h\right)\right] = 0$$

How to count DOF ?

$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) + \frac{m^2}{2} \left(h_{\mu\nu} - g_{\mu\nu} h \right)$$

Kinetic operator

Cosmological constant

Mass term

Come

Eins

Analogous to $\partial_\nu \partial_\mu F^{\mu\nu} = 0$

In Maxwell and Proca theories

This implies the (Bi...) offshell identities

$$\nabla^\mu \left[\mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) \right] = 0$$

Results in an the off-shell identity

$$\nabla^{\mu} E_{\mu\nu} = \frac{m^2}{2} \left(\nabla^{\mu} h_{\mu\nu} - g^{\rho\sigma} \nabla_{\nu} h_{\rho\sigma} \right)$$

Results in an the off-shell identity

$$\nabla^\mu E_{\mu\nu} = \frac{m^2}{2}\left(\nabla^\mu h_{\mu\nu} - g^{\rho\sigma}\nabla_\nu h_{\rho\sigma}\right)$$

And the on-shell relation

$$\nabla^\mu h_{\mu\nu} - \nabla_\nu h \simeq 0$$

4 vector
constraints

Kills 4 out of 10 DOF of $h_{\mu\nu}$

Results in an the off-shell i

$$\nabla^{\mu}$$

Analogous to the constraint

$$m^2 \partial_\nu A^\nu \simeq 0$$

of Proca theory

And the on-shell relation

$$\nabla^{\mu} h_{\mu\nu} - \nabla_\nu h \simeq 0$$

4 vector constraints

Kills 4 out of 10 DOF of $h_{\mu\nu}$

Taking an extra derivative of the field equation operator yields (off shell)

$$\nabla^\mu \nabla^\nu E_{\mu\nu} = \frac{m^2}{2} \left(\nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^2 h \right)$$

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While tracing it with the metric gives

$$g^{\mu\nu} E_{\mu\nu} = \nabla^2 h - \nabla^\mu \nabla^\nu h_{\mu\nu} + \left(\Lambda - \frac{3m^2}{2} \right) h$$

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$$g^{\mu\nu} E_{\mu\nu} = \nabla^2 h - \nabla^\mu \nabla^\nu h_{\mu\nu} + \left(\Lambda - \frac{3m^2}{2} \right) h$$

Hence we have the identity

$$2\nabla^\mu \nabla^\nu E_{\mu\nu} + m^2 g^{\mu\nu} E_{\mu\nu} = \frac{m^2}{2} \left(2\Lambda - 3m^2 \right) h$$

Yielding on shell $\boxed{\left(2\Lambda - 3m^2 \right) h \simeq 0}$

$$\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$$

Generically: yields $h \simeq 0$

i.e. a “scalar constraint” $\mathcal{C} \equiv h \simeq 0$

reducing from 6 to 5 the

number of propagating DOF

$$\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$$

However, if $2\Lambda = 3m^2$

Then this vanishes identically

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However, if $2\Lambda = 3m^2$

Then this vanishes identically

As is $2\nabla^\mu \nabla^\nu E_{\mu\nu} + m^2 g^{\mu\nu} E_{\mu\nu}$

Shows the existence of a gauge symmetry

$$\Delta h_{\mu\nu} = \left(\nabla_\mu \nabla_\nu + \frac{m^2}{2} g_{\mu\nu}\right) \xi(x) = \left(\nabla_\mu \nabla_\nu + \frac{\Lambda}{3} g_{\mu\nu}\right) \xi(x)$$

Hence, if $2\Lambda = 3m^2$

one has 6 - 2 = 4 DOF

The massive graviton is said to be "Partially massless" (PM)

Two questions:

Can a fully non linear PM theory exist ?

Can a PM graviton exist on non Einstein space-times ?

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Can a PM graviton exist on non Einstein space-times ?

Here we address the second one…

First, we need to introduce the theory of a massive graviton on arbitrary backgrounds

2. Consistent massive graviton on arbitrary backgrounds

$$S^{(2)} = -\frac{1}{2} M_g^2 \int \mathrm{d}^4 x \sqrt{|g|} h_{\mu\nu} \left(\tilde{\mathcal{E}}^{\mu\nu\rho\sigma} + m^2 \mathcal{M}^{\mu\nu\rho\sigma} \right) h_{\rho\sigma}$$

Einstein-Hilbert kinetic operator

Mass term

The theory has been obtained in

L.Bernard, CD, M. von Strauss
1410.8302 + 1504.04382
+ 1512.03620 (with A. Schmidt-May)

out of the dRGT theory

de Rahm, Gabadadze;
de Rham, Gababadze, Tolley 2010, 2011

Fully non linear theory for a dynamical metric + a non dynamical one which has been shown to propagate 5 (or less d.o.f.) in a fully non linear way …

… evading Boulware-Deser no-go « theorem »

Our massive graviton theory is defined by

Our massive graviton theory is defined by

1. A symmetric tensor $S_{\mu\nu}$ obtained from the background curvature solving

$$R^{\mu}{}_{\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2} e_1 \beta_1 \right) \delta^{\mu}_{\nu} + \left(\beta_1 + \beta_2 e_1 \right) S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu} \right]$$

with β_0, β_1 and β_2 dimensionless parameters and *m* the graviton mass, e_i the symmetric polynomials

$$\left\{ \begin{aligned} e_0 &= 1\,, \\ e_1 &= S^{\rho}{}_{\rho}\,, \\ e_2 &= \frac{1}{2}\left(S^{\rho}{}_{\rho} S^{\nu}{}_{\nu} - S^{\rho}{}_{\nu} S^{\nu}{}_{\rho} \right)\,, \\ e_3 &= \frac{1}{6}\left(S^{\rho}{}_{\rho} S^{\nu}{}_{\nu} S^{\mu}{}_{\mu} - 3 S^{\mu}{}_{\mu} S^{\rho}{}_{\nu} S^{\nu}{}_{\rho} + 2 S^{\rho}{}_{\nu} S^{\nu}{}_{\mu} S^{\mu}{}_{\rho} \right)\,, \\ e_4 &= \det(S)\,. \end{aligned} \right.$$

2. The following (linear) field equations

$$
\begin{aligned}
E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{m^2}{2} \Big[& 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right) \left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right) \\
& - \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma} \\
& - \left(\beta_1 + \beta_2 e_1\right) \left(g_{\mu\rho} \delta S^{\rho}{}_{\nu} + g_{\nu\rho} \delta S^{\rho}{}_{\mu}\right) \Big] \simeq 0 \,,
\end{aligned}
$$

2. The following (linear) field equations

$$E_{\mu\nu} \equiv \boxed{\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma}} + \frac{m^2}{2} \Big[2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right) \left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right)$$
$$- \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma}$$
$$- \left(\beta_1 + \beta_2 e_1\right) \left(g_{\mu\rho} \delta S^{\rho}{}_{\nu} + g_{\nu\rho} \delta S^{\rho}{}_{\mu}\right) \Big] \simeq 0 \,,$$

Linearized Einstein operator

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \equiv -\frac{1}{2} \Big[\delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} \nabla^2 + g^{\rho\sigma} \nabla_{\mu} \nabla_{\nu} - \delta^{\rho}_{\mu} \nabla^{\sigma} \nabla_{\nu} - \delta^{\rho}_{\nu} \nabla^{\sigma} \nabla_{\mu} - g_{\mu\nu} g^{\rho\sigma} \nabla^2 + g_{\mu\nu} \nabla^{\rho} \nabla^{\sigma}$$
$$+ \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} R - g_{\mu\nu} R^{\rho\sigma} \Big] h_{\rho\sigma} \,,$$

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\begin{aligned}
E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{m^2}{2}\Big[& 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right)\left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right) \\
& - \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma} \\
& - \left(\beta_1 + \beta_2 e_1\right)\left(g_{\mu\rho} \boxed{\delta S^{\rho}{}_{\nu}} + g_{\nu\rho} \delta S^{\rho}{}_{\mu}\right)\Big] \simeq 0 \,,
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\lambda}{}_{\mu} = \frac{1}{2} g^{\nu\lambda}\Big[& e_4\, c_1 \left(\delta^{\rho}_{\nu}\delta^{\sigma}_{\mu} + \delta^{\sigma}_{\nu}\delta^{\rho}_{\mu} - g_{\mu\nu} g^{\rho\sigma}\right) + e_4\, c_2 \left(S^{\rho}_{\nu}\delta^{\sigma}_{\mu} + S^{\sigma}_{\nu}\delta^{\rho}_{\mu} - S_{\mu\nu} g^{\rho\sigma} - g_{\mu\nu} S^{\rho\sigma}\right) \\
& - e_3\, c_1 \left(\delta^{\rho}_{\nu} S^{\sigma}_{\mu} + \delta^{\sigma}_{\nu} S^{\rho}_{\mu}\right) + \left(e_2\, c_1 - e_4\, c_3 + e_3\, c_2\right) S_{\mu\nu} S^{\rho\sigma} \\
& + e_4\, c_3 \left[\delta^{\sigma}_{\mu}[S^2]^{\rho}_{\nu} + \delta^{\rho}_{\mu}[S^2]^{\sigma}_{\nu} - g^{\rho\sigma}[S^2]_{\mu\nu} + \delta^{\rho}_{\nu}[S^2]^{\sigma}_{\mu} + \delta^{\sigma}_{\nu}[S^2]^{\rho}_{\mu} - g_{\mu\nu}[S^2]^{\rho\sigma}\right] \\
& - e_3\, c_2 \left(S^{\rho}_{\nu} S^{\sigma}_{\mu} + S^{\sigma}_{\nu} S^{\rho}_{\mu}\right) - e_3\, c_3 \left(S^{\sigma}_{\mu}[S^2]^{\rho}_{\nu} + S^{\rho}_{\mu}[S^2]^{\sigma}_{\nu} + S^{\rho}_{\nu}[S^2]^{\sigma}_{\mu} + S^{\sigma}_{\nu}[S^2]^{\rho}_{\mu}\right) \\
& + \left(e_3\, c_3 - e_1\, c_1\right) \left(S^{\rho\sigma}[S^2]_{\mu\nu} + S_{\mu\nu}[S^2]^{\rho\sigma}\right) - \left(c_1 - e_2\, c_3\right) \left([S^2]^{\rho}_{\nu}[S^2]^{\sigma}_{\mu} + [S^2]^{\sigma}_{\nu}[S^2]^{\rho}_{\mu}\right) \\
& + c_4\, [S^2]_{\mu\nu}[S^2]^{\rho\sigma} + c_1 \left([S^3]_{\mu\nu} S^{\rho\sigma} + S_{\mu\nu}[S^3]^{\rho\sigma}\right) + c_2 \left([S^3]_{\mu\nu}[S^2]^{\rho\sigma} + [S^2]_{\mu\nu}[S^3]^{\rho\sigma}\right) \\
& + c_3\, [S^3]_{\mu\nu}[S^3]^{\rho\sigma} \Big] h_{\rho\sigma} \,,
\end{aligned}
$$

DOF counting ?

We follow the analysis for the case of Einstein space-times:

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The linearized Bianchi identity $$\nabla^{\mu}\mathcal{G}_{\mu\nu} = 0$$

Yields 4 vector constaints reducing from 10 to 6 the number of propagating DOF

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A scalar constraint $\mathcal{C} \simeq 0$ reduces to 5 the number of DOF

$$\mathcal{C} \equiv (S^{-1})^{\nu}{}_{\rho}\nabla^{\rho}\nabla^{\mu}E_{\mu\nu} + \frac{m^2\beta_1}{2}g^{\mu\nu}E_{\mu\nu} + m^2\beta_2 S^{\mu\nu}E_{\mu\nu} \simeq 0$$

3. PM graviton on non Einstein space-times ?

We look for (non Einstein) space-times where the scalar constraint

$$\mathcal{C} \equiv (S^{-1})^{\nu}{}_{\rho} \nabla^{\rho} \nabla^{\mu} E_{\mu\nu} + \frac{m^2 \beta_1}{2} g^{\mu\nu} E_{\mu\nu} + m^2 \beta_2 S^{\mu\nu} E_{\mu\nu} \simeq 0$$

Identically vanishes

[Yielding the gauge symmetry $h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu}$ with

$$\Delta h_{\mu\nu} = \left[(S^{-1})_{\mu}{}^{\rho} \nabla_{\rho} \nabla_{\nu} + (S^{-1})_{\nu}{}^{\rho} \nabla_{\rho} \nabla_{\mu} + m^2 \beta_1 g_{\mu\nu} + 2m^2 \beta_2 S_{\mu\nu} \right] \xi(x)$$
]

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$\Delta h_{\mu\nu} = \left[(S^{-1})_{\mu}{}^{\rho} \nabla_{\rho} \nabla_{\nu} + (S^{-1})_{\nu}{}^{\rho} \nabla_{\rho} \nabla_{\mu} + m^2 \beta_1 g_{\mu\nu} + 2m^2 \beta_2 S_{\mu\nu} \right] \xi(x)$]

We need to look in detail at the structure of the constraint

The scalar constraint reads

$$\mathcal{C} = m^2 \left[\left(A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \right) \tilde{h}_{\beta\lambda} + B_{\rho}^{\beta\lambda} \, \nabla^{\rho} \tilde{h}_{\beta\lambda} \right]$$

With $h_{\mu\nu} = \left(S^{\lambda}{}_{\mu} \delta^{\beta}_{\nu} + S^{\lambda}{}_{\nu} \delta^{\beta}_{\mu} \right) \tilde{h}_{\beta\lambda}$

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With $h_{\mu\nu} = \Big(S^\lambda{}_\mu \delta^\beta_\nu + S^\lambda{}_\nu \delta^\beta_\mu \Big) \tilde{h}_{\beta\lambda}$

$$A^{\beta\lambda} \equiv m^2 \, S^\beta{}_\rho \Big[\Big(\beta_0\beta_1 + \beta_0\beta_2 e_1 + \frac{1}{2}\beta_1^2 e_1 \Big) g^{\rho\lambda} + \Big(-2\beta_0\beta_2 - \frac{1}{2}\beta_1^2 - 2\beta_2^2 e_2 + \beta_2^2 e_1^2 \Big) S^{\rho\lambda} - \beta_2^2 e_1 [S^2]^{\rho\lambda} \Big] \, ,$$

The scalar constraint reads

$$\mathcal{C} = m^2\Big[\Big(A^{\beta\lambda} + \tilde{A}^{\beta\lambda}\Big)\,\tilde{h}_{\beta\lambda} + B^{\beta\lambda}_{\rho}\,\nabla^{\rho}\tilde{h}_{\beta\lambda}\Big]$$

With $h_{\mu\nu} = \Big(S^{\lambda}{}_{\mu}\delta^{\beta}_{\nu} + S^{\lambda}{}_{\nu}\delta^{\beta}_{\mu}\Big)\tilde{h}_{\beta\lambda}$

$$\begin{aligned}
\tilde{A}^{\beta\lambda} \equiv\ & \frac{1}{2}\left(\beta_1 + \beta_2 e_1\right)[S^{-1}]^{\nu}{}_{\gamma}\Big[-\nabla^{\gamma}S^{\rho\lambda}\nabla_{\nu}S^{\beta}_{\rho} + \nabla^{\gamma}S^{\beta}_{\rho}\nabla^{\lambda}S^{\rho}_{\nu} + \nabla^{\gamma}S^{\rho}_{\nu}\nabla^{\lambda}S^{\beta}_{\rho} - \nabla^{\gamma}S_{\rho\nu}\nabla^{\rho}S^{\beta\lambda} \\
& - S^{\rho\lambda}\nabla^{\gamma}\nabla_{\nu}S^{\beta}_{\rho} + S^{\beta}_{\rho}\nabla^{\gamma}\nabla^{\lambda}S^{\rho}_{\nu}\Big] + \beta_2\,[S^{-1}]^{\nu}{}_{\gamma}\Big[S^{\beta}_{\rho}\nabla^{\lambda}S^{\rho}_{\nu}\nabla^{\gamma}e_1 - S^{\beta}_{\rho}\nabla_{\nu}S^{\rho\lambda}\nabla^{\gamma}e_1 \\
& + S^{\lambda}_{\rho}\nabla^{\gamma}S^{\beta}_{\mu}\nabla_{\nu}S^{\rho\mu} + S^{\beta}_{\mu}\nabla^{\gamma}S^{\lambda}_{\rho}\nabla_{\nu}S^{\rho\mu} + S^{\lambda}_{\mu}\nabla^{\gamma}S^{\mu\rho}\nabla_{\nu}S^{\beta}_{\rho} + S^{\mu\rho}\nabla^{\gamma}S^{\lambda}_{\mu}\nabla_{\nu}S^{\beta}_{\rho} \\
& - 2S^{\beta}_{\mu}\nabla^{\gamma}S^{\mu\lambda}\nabla_{\nu}e_1 - S^{\rho}_{\mu}\nabla^{\gamma}S^{\mu}_{\nu}\nabla^{\beta}S^{\lambda}_{\rho} - S^{\beta}_{\mu}\nabla^{\gamma}S^{\mu}_{\rho}\nabla^{\lambda}S^{\rho}_{\nu} - S^{\mu}_{\rho}\nabla^{\gamma}S^{\beta}_{\mu}\nabla^{\lambda}S^{\rho}_{\nu} \\
& - S^{\beta}_{\rho}\nabla^{\gamma}S^{\mu}_{\nu}\nabla^{\lambda}S^{\rho}_{\mu} + S^{\beta}_{\mu}\nabla^{\gamma}S^{\mu}_{\nu}\nabla^{\lambda}e_1 + S^{\mu}_{\rho}\nabla^{\gamma}S_{\mu\nu}\nabla^{\rho}S^{\beta\lambda} - S^{\lambda}_{\mu}\nabla^{\gamma}S^{\mu}_{\nu}\nabla^{\rho}S^{\beta}_{\rho} \\
& - S^{\lambda}_{\mu}\nabla^{\gamma}S^{\beta}_{\rho}\nabla^{\mu}S^{\rho}_{\nu} - S^{\beta}_{\rho}\nabla^{\gamma}S^{\lambda}_{\mu}\nabla^{\mu}S^{\rho}_{\nu} - S^{\lambda}_{\mu}\nabla^{\gamma}S^{\rho}_{\nu}\nabla^{\mu}S^{\beta}_{\rho} + 2S^{\beta}_{\mu}\nabla^{\gamma}S^{\mu\lambda}\nabla^{\rho}S_{\rho\nu} \\
& + 2S^{\beta}_{\rho}\nabla^{\gamma}S_{\mu\nu}\nabla^{\mu}S^{\rho\lambda} + S^{\lambda}_{\rho}S^{\beta}_{\mu}\nabla^{\gamma}\nabla_{\nu}S^{\rho\mu} + [S^2]^{\lambda\rho}\nabla^{\gamma}\nabla_{\nu}S^{\beta}_{\rho} - [S^2]^{\beta\lambda}\nabla^{\gamma}\nabla_{\nu}e_1 \\
& - [S^2]^{\beta}_{\rho}\nabla^{\gamma}\nabla^{\lambda}S^{\rho}_{\nu} - S^{\lambda}_{\mu}S^{\beta}_{\rho}\nabla^{\gamma}\nabla^{\mu}S^{\rho}_{\nu} + [S^2]^{\beta\lambda}\nabla^{\gamma}\nabla^{\rho}S_{\rho\nu}\Big] + \beta_2\Big[+\nabla^{\beta}S^{\lambda}_{\gamma}\nabla^{\gamma}e_1 \\
& - \nabla_{\gamma}S^{\beta\lambda}\nabla^{\gamma}e_1 - \nabla^{\mu}S^{\rho}_{\mu}\nabla^{\beta}S^{\lambda}_{\rho} - \nabla^{\mu}S^{\beta}_{\rho}\nabla^{\lambda}S^{\rho}_{\mu} + \nabla^{\mu}S^{\beta}_{\mu}\nabla^{\lambda}e_1 + \nabla_{\mu}S^{\mu}_{\rho}\nabla^{\rho}S^{\beta\lambda} \\
& - \nabla^{\mu}S^{\lambda}_{\mu}\nabla^{\rho}S^{\beta}_{\rho} - \nabla^{\rho}S^{\lambda}_{\mu}\nabla^{\mu}S^{\beta}_{\rho} + 2\nabla_{\mu}S^{\beta}_{\rho}\nabla^{\mu}S^{\rho\lambda} - S^{\beta}_{\rho}\nabla^{\lambda}\nabla^{\mu}S^{\rho}_{\mu} + S^{\beta}_{\gamma}\nabla^{\gamma}\nabla^{\lambda}e_1 \\
& - S^{\lambda}_{\gamma}\nabla^{\gamma}\nabla^{\rho}S^{\beta}_{\rho} + S^{\beta}_{\rho}\nabla^{\gamma}\nabla_{\gamma}S^{\rho\lambda}\Big] + (\beta \leftrightarrow \lambda)\,,
\end{aligned}$$

The scalar constraint reads

$$\mathcal{C} = m^2 \left[\left(A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \right) \tilde{h}_{\beta\lambda} + B_\rho^{\beta\lambda} \nabla^\rho \tilde{h}_{\beta\lambda} \right]$$

With $h_{\mu\nu} = \left(S^\lambda{}_\mu \delta^\beta_\nu + S^\lambda{}_\nu \delta^\beta_\mu \right) \tilde{h}_{\beta\lambda}$

$$\begin{aligned}
B_\rho^{\beta\lambda} \equiv \frac{1}{2}\left(\beta_1 + \beta_2 e_1\right)[S^{-1}]^\nu_\gamma \; & \left[-S^{\sigma\lambda}\delta^\gamma_\rho \nabla_\nu S^\beta_\sigma + \delta^\gamma_\rho S^\beta_\sigma \nabla^\lambda S^\sigma_\nu + \delta^\lambda_\rho S^\beta_\sigma \nabla^\gamma S^\sigma_\nu - S^{\beta\lambda}\nabla^\gamma S_{\nu\rho}\right] \\
& + \beta_2 [S^{-1}]^\nu{}_\gamma \Big[\delta^\gamma_\rho S^\lambda_\delta S^\beta_\mu \nabla_\nu S^{\delta\mu} + \delta^\gamma_\rho [S^2]^{\lambda\mu} \nabla_\nu S^\beta_\mu - \delta^\gamma_\rho [S^2]^{\beta\lambda} \nabla_\nu e_1 - \delta^\gamma_\rho [S^2]^\beta_\mu \nabla^\lambda S^\mu_\nu \\
& - \delta^\gamma_\rho S^\lambda_\mu S^\beta_\delta \nabla^\mu S^\delta_\nu + \delta^\gamma_\rho [S^2]^{\beta\lambda} \nabla^\mu S_{\mu\nu} + S^{\beta\lambda} S^\mu_\rho \nabla^\gamma S_{\mu\nu} + [S^2]^{\beta\lambda} \nabla^\gamma S_{\rho\nu} - \delta^\beta_\rho [S^2]^\lambda_\mu \nabla^\gamma S^\mu_\nu \\
& - S^\beta_\rho S^\lambda_\mu \nabla^\gamma S^\mu_\nu \Big] + \beta_2 \Big[-S^\beta_\delta \nabla^\lambda S^\delta_\rho + S^\beta_\rho \nabla^\lambda e_1 - S^\lambda_\mu \nabla^\mu S^\beta_\rho + 2 S^\beta_\delta \nabla_\rho S^{\delta\lambda} + \delta^\beta_\rho S^\lambda_\gamma \nabla^\gamma e_1 \\
& - S^{\beta\lambda} \nabla_\rho e_1 + S^{\beta\lambda} \nabla_\mu S^\mu_\rho - \delta^\beta_\rho S^\lambda_\delta \nabla^\mu S^\delta_\mu - S^\beta_\rho \nabla^\mu S^\lambda_\mu \Big] + (\beta \leftrightarrow \lambda)\,.
\end{aligned}$$

To get a PM theory we need to look for space-times

where $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$ vanish identically.

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Most general solution ?

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Most general solution ?

Assume $\boxed{\nabla_{\rho} S_{\mu\nu} = 0}$

(i.e. $S_{\mu\nu}$ covariantly constant)

makes $\tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$ vanish.

Space-times possessing a covariantly constant tensor $H_{\mu\nu}$ are classified as (provided $H_{\mu\nu}$ is not proportional to the metric)

1. Spacetime is $2 \otimes 2$ decomposable

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{ab}(x^c)\mathrm{d}x^a \mathrm{d}x^b + g_{ij}(x^k)\mathrm{d}x^i \mathrm{d}x^j$$

and $H_{\mu\rho}H^\rho{}_\nu = H_{\mu\nu}$

2. The spacetime admits a covariantly constant vector N^μ and $H_{\mu\nu} = N_\mu N_\nu$

Note further that the definition

$$R^{\mu}{}_{\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2} e_1 \beta_1 \right) \delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu} \right]$$

Imposes that $\nabla_\rho S_{\mu\nu} = 0 \Rightarrow \nabla_\rho R_{\mu\nu} = 0$

Hence the space-time must be "Ricci Symmetric"

…i.e. the covariantly constant tensor is the Ricci tensor…

The spacetimes of interest for us here will all have

$$(R^2)^{\rho}{}_{\nu} = r_1\, R^{\rho}{}_{\nu} + r_2\, \delta^{\rho}_{\nu}$$ with r_1 and r_2 constant

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\begin{cases} \delta^{\beta}_{\lambda}\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2} e_1\right) + S^{\beta}_{\lambda}\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2}\right) - (S^2)^{\beta}_{\lambda}\left(e_1\beta_2^2\right) = 0 \\ R^{\mu}{}_{\nu} = m^2\left[\left(\beta_0 + \frac{1}{2} e_1\beta_1\right)\delta^{\mu}_{\nu} + \left(\beta_1 + \beta_2 e_1\right) S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu}\right] \end{cases}$$

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In order to get a vanishing $\mathcal{C} = m^2 A^{\beta\lambda}\tilde{h}_{\beta\lambda}$

The spacetimes of interest for us here will all have

$$(R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu \quad \text{with } r_1 \text{ and } r_2 \text{ constant}$$

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\begin{cases} \delta^\beta_\lambda\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \dfrac{\beta_1^2}{2} e_1\right) + S^\beta_\lambda\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \dfrac{\beta_1^2}{2}\right) - (S^2)^\beta_\lambda(e_1\beta_2^2) = 0 \\ \boxed{R^\mu{}_\nu = m^2\left[\left(\beta_0 + \dfrac{1}{2}e_1\beta_1\right)\delta^\mu_\nu + (\beta_1 + \beta_2 e_1)\, S^\mu{}_\nu - \beta_2 (S^2)^\mu{}_\nu\right]} \end{cases}$$

From the definition of $S^\rho{}_\nu$

The spacetimes of interest for us here will all have

$(R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu$ with r_1 and r_2 constant

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\begin{cases} \delta^\beta_\lambda\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \dfrac{\beta_1^2}{2} e_1\right) + S^\beta_\lambda\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \dfrac{\beta_1^2}{2}\right) - (S^2)^\beta_\lambda (e_1\beta_2^2) = 0 \\ R^\mu{}_\nu = m^2\left[\left(\beta_0 + \dfrac{1}{2} e_1\beta_1\right)\delta^\mu_\nu + (\beta_1 + \beta_2 e_1)\, S^\mu{}_\nu - \beta_2 (S^2)^\mu{}_\nu\right] \end{cases}$$

NB: this implies that the Ricci tensor obeyes indeed the required relation

Explicit solutions (I)

with
$$\begin{cases} (R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu \\ u \equiv \dfrac{\beta_0\beta_2}{\beta_1^2}, \quad v \equiv m^2\beta_0 \dfrac{16u^2 - 24u - 3}{u\,(4u+3)} \end{cases}$$

dimensionless dimensionful

$$2 \otimes 2 \begin{cases} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \dfrac{2\mathrm{d}x^0\mathrm{d}x^1}{(1+(R-E)x^0x^1/8)^2} - \dfrac{2\mathrm{d}x^2\mathrm{d}x^3}{(1-(R+E)x^2x^3/8)^2} \\ \text{type D}: \quad r_1 = \dfrac{R}{2}\,, \qquad r_2 = -\dfrac{1}{16}\left(R^2 - E^2\right) \\ E^2 = \dfrac{72\left(9+60u+144u^2+64u^3\right)}{\left(-9-132u-48u^2+64u^3\right)^2} R^2 \\ v = \left(2 + \dfrac{96u}{-9-132u-48u^2+64u^3}\right) R \end{cases}$$

Analogous to $2\Lambda = 3m^2$

Explicit solutions (II)

with
$$\begin{cases} (R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu \\ u \equiv \dfrac{\beta_0\beta_2}{\beta_1^2}, \qquad v \equiv m^2\beta_0 \dfrac{16u^2 - 24u - 3}{u\,(4u+3)} \end{cases}$$

dimensionless

dimensionful

$$1 \otimes 3 \begin{cases} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \dfrac{\mathrm{d}x^2 + \mathrm{d}y^2 - \epsilon\,\mathrm{d}w^2}{\left[1 + R\,(x^2+y^2-\epsilon\, w^2)/24\right]^2} + \epsilon\,\mathrm{d}z^2 \\ \text{type O}: \quad r_1 = \dfrac{R}{3}, \qquad r_2 = 0 \\ 256u^4 + 1280u^3 + 4320u^2 + 144u - 135 = 0 \\ v = \dfrac{6\,(32u^3+42u+9)}{128u^3+144u^2+144u+27} R \end{cases}$$

Analogous to $2\Lambda = 3m^2$

One simple example of the last kind is Einstein static Universe !

$$\begin{cases} \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2\,\mathrm{d}\Sigma^2 \\ \text{with } \mathrm{d}\Sigma^2 = \gamma_{ij}\,\mathrm{d}x^i \mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-k\,r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \end{cases}$$

$$\Rightarrow \quad R = \frac{6k}{a^2}$$

One simple example of the last kind is Einstein static Universe !

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$$R = \frac{6k}{a^2}$$

The gauge symmetry reads

$$\left\{\begin{array}{l} \Delta h_{tt} = \left[-\frac{(4u+3)(v-2R)}{v}\nabla_t\partial_t + \frac{u(6R+8Ru-3v)v}{(-3-24u+16u^2)(-2R+vv)}\right]\xi(x)\,, \\ \Delta h_{ti} = -\frac{(4u+3)(v-2R)\left(\frac{8R}{3}-2v\right)}{2v\left(\frac{8R}{3}-v\right)}\nabla_{(t}\partial_{i)}\,\xi(x)\,, \\ \Delta h_{ij} = \left[-\frac{(4u+3)(v-2R)}{\frac{8R}{3}-v}\nabla_i\partial_j + \frac{uv(-18R+8Ru+9v)}{3(-3-24u+16u^2)(-2R+v)}\,a^2\,\gamma_{ij}\right]\xi(x)\,, \end{array}\right.$$

With e.g. one solution being $(u,\,v) = (-0.2006,\,0.6622R)$

Conclusions

PM exists on non Einstein spacetimes !

(in contrast with previous no-go claim by Deser, Joung, Waldron
In 1208.1307 [hep-th]…
… in particular some of our solution have a vanishing Bach tensor)

Solution for the vanishing
Of $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$
are not known in full generality !