Quantum field theory based on point particles well describes electroweak and nuclear strong forces but it is inconsistent for gravity.

Supersymmetry and strings are hoped to provide a consistent theory of quantum gravity but they require branes and very little is known about their quantum properties.

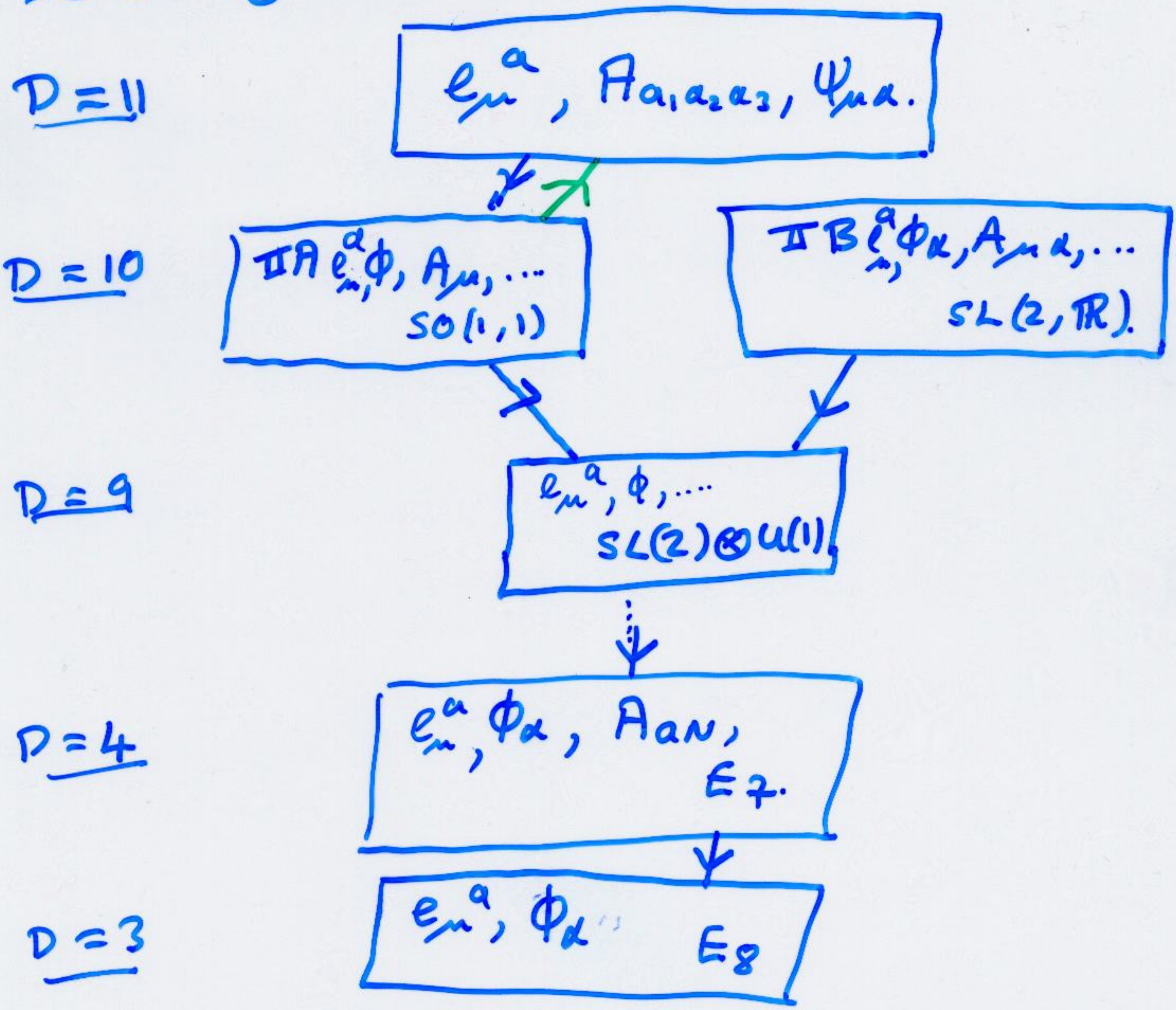
Finding some of the symmetries of the theory of strings and branes is a first step.

Superstrings are a perturbatively consistent theory of quantum gravity. However, their non-perturbative effects are difficult to compute.

Supergravity theories encode the perturbative and non-perturbative effects of string theory at low energy.

– They have E_n symmetries and require branes as well as strings

M-theory and the maximal supergravities



- In D dimensions one finds an E_{11-D} symmetry
- M theory is just the set of connections between the different theories.

In addition to the maximal supergravity theories in $D = 2, 3, \ldots, 9, 11$ with two in $D = 10$. we have gauged supergravities

- introduce a cosmological constant $\Lambda \int d^d x \det e$ and add terms so as to preserve all supersymmetries

- None in $D = 11$ and IIB but one in IIA (Romans).

- For $D < 10$ there are many such theories which gauge part of the E_n rigid symmetry

Non-linear realizations of $G \otimes_s \ell$.

Let $G \otimes_s \ell$ be the semidirect product of G and its representation ℓ.

$$[R^\alpha, R^\beta] = f^{\alpha\beta}{}_\gamma R^\gamma$$

$$[R^\alpha, \ell_A] = -(D^\alpha)_A{}^B \ell_B$$

can take ℓ_A's to commute.

Example $G = SO(1,3)$, $\ell = \{P_a\}$.

$$[J_{ab}, P_c] = -\eta_{ac} P_b + \eta_{bc} P_a.$$

The non-linear realization of $G \otimes_s \ell$ with local subgroup H is built from $g \in G \otimes_s \ell$ which can be written as

$$g = g_\ell \, g_\epsilon = e^{z^A \ell_A} e^{A_\alpha R^\alpha}$$

The dynamics is invariant under

$$\underset{\substack{\in G \\ \text{rigid}}}{g \to g_0 \, g} \quad ; \quad \underset{\substack{\in H \\ \text{local} \\ (\text{depends on } z^A)}}{g \to g \, h}$$

Example 1.

If $G = H$ then $g = e^{z^A \ell_A}$

* If G = Poincaré, H = Lorentz then

$g = e^{x^a P_a}$ ⇝ Minkowski space

$$x^{a\prime} = \Lambda^a{}_b x^b + \varepsilon^a.$$

* G = super Poincaré, H = Lorentz

$$g = e^{x^a P_a} e^{\theta^\alpha P_\alpha}$$

⇝ superspace

Salam
Strathdee.

Example 2 no ℓ $\rightsquigarrow$ $g = e^{A_\alpha R^\alpha}$
but let A_α depend on an introduced
spacetime. (ccwz)

The Cartan forms $\mathcal{V} = g^{-1}dg$ are
inert under rigid transformations
but under local.

$$\mathcal{V} \to h^{-1}\mathcal{V}h + h^{-1}dh.$$

We can write

$$\mathcal{V} = \underset{\text{coset.}}{P} + \underset{\in H}{Q}$$

As $P \to h^{-1}Ph$ an invariant action is

$$\int d^dx \, \mathrm{Tr}\, P^2.$$

- $G = SU(2) \times SU(2)$, $H = SU(2)_{\text{diag}}$

π - dynamics

- The scalars in supergravity

- $D = 4$ $G = E_7$, $H = SU(8)$

$\rightsquigarrow$ $133 - 63 = 70$ scalars.

- IIB, $G = SL(2)$, $H = SO(2)$

Two scalars ϕ, χ.

when a rigid symmetry G is spontaneously broken to H there are $\dim G - \dim H$ massless (Goldstone) particles whose low energy dynamics is "the non-linear realization of G with local subgroup H."

π dynamics	supergravity
$SU(2) \otimes SU(2)$	E_{11}
quarks	?

The E_n symmetries in the maximal supergravity theories were thought to be a quirk of dimensional reduction.

In $D = 4$ the scalars belong to the non-linear realisation of E_7 with local subgroup $SU(8)$.

$$\phi_\alpha \longrightarrow R^\alpha \in E_7 .$$

Can we generalise this to all fields. In $D = 11$ this means

$$\left.\begin{array}{l} h_a{}^b \longleftrightarrow K^a{}_b \\ A_{a_1 a_2 a_3} \longleftrightarrow R^{a_1 a_2 a_3} \\ \quad\vdots \end{array}\right\} E_{11}$$

we need an infinite number of fields.

Kac-Moody Algebras.

Lie algebras $\rightsquigarrow$ $[H_i, H_j] = 0$

$[H_i, E_\alpha] = \alpha_i E_\alpha$. $\forall$ roots α.

$+ \cdots$

$\rightsquigarrow$ simple roots α_a $\rightsquigarrow$ Cartan matrix

$$A_{ab} = 2\frac{(\alpha_a, \alpha_b)}{(\alpha_a, \alpha_a)}.$$

$\rightsquigarrow$ Dynkin diagram.

All Lie algebras in the Cartan lists had

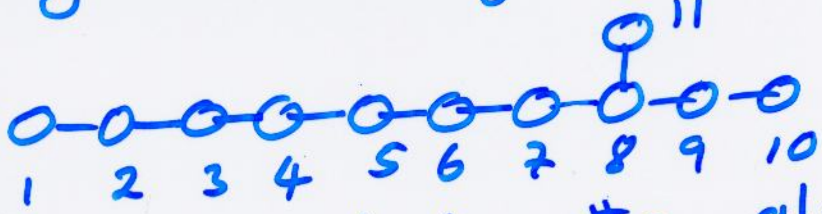
- A_{ab} -ve integer if $a \neq b$
- $A_{ab} = 0 \iff A_{ba} = 0$
- $v^a A_{ab} v^b > 0 \quad \forall\, v^a$.

Serre found a way to go from Cartan matrix to the Lie algebra which is generated by $\{E_a, F_a, H_a\}_{a=1,\ldots,r}$ subject to Serre relations.

Kac and Moody just dropped the last condition.

The $E_{11}{}^{10}$ algebra (0104 081)

The Dynkin diagram



completely specifies the algebra.
Decomposing into representations of $SL(11)$ we have

$$R^\alpha = \{ \ldots, K^a{}_b, R^{a_1a_2a_3}, R^{a_1\ldots a_6}, R^{a_1\ldots a_8, b}, \ldots \}$$

The level $= \frac{1}{3}$(no of up - down indices).

The algebra is known up to level ± 4

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b$$

$$\vdots$$

$$[R^{a_1a_2a_3}, R^{a_4a_5a_6}] = 2 R^{a_1\ldots a_6}.$$

The "vector" representation (0307098)

The first fundamental representation has highest weight Λ_1 with $(\Lambda_1, \alpha_a) = \delta_{a1}$

Decomposed into representations of $SL(11)$ we find

$$l_A = \{ P_a,\ Z^{a_1 a_2},\ Z^{a_1 \ldots a_5},\ Z^{a_1 \ldots a_7, b},$$
$$Z^{a_1 \ldots a_8},\ \ldots \}$$

Encodes all brane charges.

The algebra $E_{11} \otimes_s l_1$ is

$$[K^a{}_b, P_c] = -\delta^a_c P_b + \tfrac{1}{2} \delta^a_b P_c$$

$$\vdots$$

$$[R^{a_1 a_2 a_3}, P_c] = 3\, \delta^{[a_1}_{b}\, Z^{a_2 a_3]}$$

$$\vdots$$

Non-linear Realization of $E_{11} \otimes l_1$ (0307098)

is constructed from $g = g_l g_E$ where

$$g_E = e^{A_{\underline{\alpha}} R^{\underline{\alpha}}} = \dots \, e^{h_a{}^b K^a{}_b} \, e^{A_{a_1 \dots a_3} R^{a_1 \dots a_3}} \cdot e^{A_{a_1 \dots a_6} R^{a_1 \dots a_6}} \dots$$

and

$$g_l = e^{x^a P_a} e^{x_{a_1 a_2} Z^{a_1 a_2}} e^{x_{a_1 \dots a_5} Z^{a_1 \dots a_5}} \dots = e^{z^A l_A}.$$

The fields $h_a{}^b$, $A_{a_1 a_2 a_3}, \dots$ are functions of the coordinates of the generalized spacetime

$$x^a, \, x_{a_1 a_2}, \, x_{a_1 \dots a_5}, \dots$$

Thus we find the fields of supergravity and the usual spacetime plus much more.

The Cartan forms are

$$\mathcal{V} = g^{-1} dg = \underbrace{dz^{\Pi} E_{\Pi}{}^{A} l_A}_{\in\, l_1} + \underbrace{dz^{\Pi} G_{\Pi,\underline{\alpha}} R^{\underline{\alpha}}}_{\in\, E_{11}}.$$

The object $E_{\Pi}{}^{A}$ is the vielbein on the spacetime

$$E_{\Pi}{}^{A} = \begin{pmatrix} e_{\mu}{}^{a} & -3 A_{\mu b_1 b_2} & \cdots \\ 0 & e^{-1}{}_{(b_1}{}^{\mu_1} e^{-1}{}_{b_2]}{}^{\mu_2} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

The $G_{A,\underline{\alpha}} = E_A{}^{\Pi} G_{\Pi,\alpha}$ just transform under $H = I_c(E_{11})$ for example

$$\delta G_{a,b}{}^{c} = 18 \Lambda^{d_1 d_2 c} G_{a,d_1 d_2 b} - 2\delta_b^c G_{a,d_1 d_2 d_3} \Lambda^{d_1 d_2 d_3} - 3 \Lambda_{a d_1 d_2} G^{d_1 d_2}{}_{,b}{}^{c}$$

where

$$G_{a,b}{}^{c} = e_a{}^{\mu} e_b{}^{\lambda} \partial_{\mu} e_{\lambda}{}^{c} \; ; \; e_{\mu}{}^{a} = (e^{h})_{\mu}{}^{a}$$

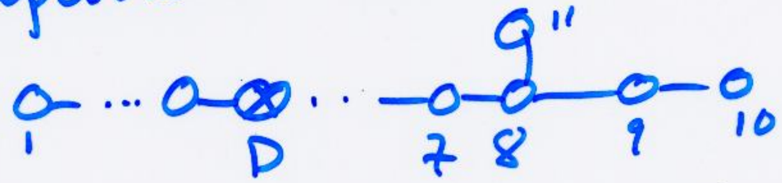
$$G_{a,b_1 b_2 b_3} = e_a{}^{\mu_1} e_{b_1}{}^{\nu_1} e_{b_2}{}^{\nu_2} e_{b_3}{}^{\nu_3} \{ \partial_{\mu_1} A_{\nu_1 \nu_2 \nu_3} \}$$

The dynamics for the low level fields $e_\mu{}^a$, $A_{a_1 a_2 a_3}$, ... is uniquely determined and when the generalized spacetime is truncated to be just the usual spacetime (x^μ) we find the equations of motion of eleven dimensional supergravity.
(1512.01644)

Conclusion

The non-linear realization of $E_{11} \otimes_s l_1$ is a unified theory. It contains

– The maximal supergravity theories in the different dimensions appear by taking a different E_{11} decomposition into $SL(D) \otimes E_{11-D}$.



– It includes all the gauged supergravities i.e. has the fields

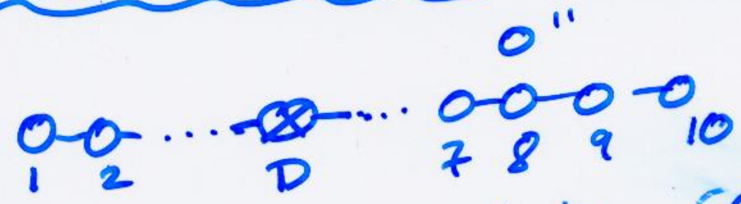
$A_{a_1 \ldots a_{D-1}} \rightsquigarrow \int d^D x \, F^2_{a_1 \ldots a_D} \det e$

$\rightsquigarrow \int d^D x \det e \, \Lambda .$

– It replaces the maximal supergravity theories as the low energy effective action for strings and branes.

– What is the physical meaning of the higher level fields and coordinates.

$E_{11} \otimes_s l_1$ in D dimensions



decompose E_{11} into $GL(D) \otimes E_{11-D}$.

ie $D = 5$ into $GL(5) \otimes E_6$.

We find the fields

$$\Phi_{\alpha}^{(78)},\ h_a{}^b,\ A_{aN}(27),\ A_{a_1a_2}{}^N(\overline{27}),\ A_{a_1a_2a_3\alpha}^{(78)}, \ldots$$

and the coordinates

$$x^a,\ x, \ \ ,$$

ie The usual fields of $D = 5$ supergravity plus their duals.

- E_{11} is a duality symmetry.
- contains a field $A_{a_1..a_4}(351)$

$\sim \int d^5x \det e \, F^2_{a_1..a_5} \sim F_{a_1..a_5} = m\,\epsilon_{a_1..a_5}$

$\sim \int d^5x \, m^2 \det e$ ie a cosmological constant.

E_{11} and brane charges.

The vector representation (l_1) of E_{11} contains

$$P_a, \; Z^{a_1 a_2}, \; Z^{a_1 \ldots a_5}, \; Z^{a_1 \ldots a_7, b}, \; \ldots \; Z^{d, c_1 \ldots c_3, e_1 \ldots e_{10}}, \; \ldots$$

point particle, M2, M5, Taub-Nut, ...

It contains all known brane charges and an ~~a~~ infinite number of new ones. that is, new branes and so new degrees of freedom.

For every element in the vector representation we find an element in the Borel subalgebra of E_{11}

$$x_a \longleftrightarrow P_a \longleftrightarrow K^a{}_b \longleftrightarrow h_a{}^b$$

$$x_{a_1 a_2} \longleftrightarrow Z^{a_1 a_2} \longleftrightarrow R^{a_1 a_2 a_3} \longleftrightarrow A_{a_1 a_2 a_3}$$

$$x_{a_1 \ldots a_5} \longleftrightarrow Z^{a_1 \ldots a_5} \longleftrightarrow R^{a_1 \ldots a_6} \longleftrightarrow A_{a_1 \ldots a_6}$$

$$\vdots$$

Einstein geometry is replaced to include higher level fields

E_{11} as a duality symmetry

Maxwell's equations, with no charges, are invariant under $\underline{E} \leftrightarrow \underline{B}$
To make this manifest we introduce A_μ and B_μ which obey.

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\kappa} \partial_\rho B_\kappa = * G_{\mu\nu}.$$

We can write this as.

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} F \\ G \end{pmatrix}$$

which has an $SO(2)$ symmetry.

E_{11} contains the fields

$$A_{a_1 \cdot a_3}, \; A_{a_1 \cdot a_6}, \; A_{a_1 \cdot a_3, b_1 \ldots b_9}, \; \ldots \; A_{a_1 \cdot a_6, b_1 \cdot b_9, c_1 \cdot c_9}, \ldots$$

They obey an infinite set of duality relations which transform into each other under E_{11}

$$F_{\mu_1 \cdot\cdot \mu_4} = \partial_{\mu_1} A_{\mu_2 \cdot\cdot \mu_4} + \cdots =$$

$$\frac{1}{7!} \epsilon_{\mu_1 \cdot\cdot \mu_4}{}^{\nu_1 \cdot\cdot \nu_7} (\partial_{\nu_1} A_{\nu_2 \cdots \nu_7} - A_{\nu_1 \cdot \nu_3} \partial_{\nu_4} A_{\nu_5 \cdots \nu_7}) + \cdots = * F_7$$

or

$$F_4 = * F_7 .$$

and

$$\partial_{[\nu_1} A_{\nu_2\nu_3\nu_4]} \doteq \epsilon_{\nu_1}{}^{\tau_1 \ldots \tau_{10}} (\partial_{\tau_1} A_{\tau_2 \ldots \tau_{10}, \nu_2\nu_3\nu_4}) + \ldots$$

etc.

Taking another derivative we find.

$$\partial^{\nu} F_{\nu\mu_1 \ldots \mu_3} = 0 = \partial^{\nu} f_{\nu\mu_1 \ldots \mu_6} = \ldots\ldots$$

The degrees of freedom can be described in an infinite number of ways

The same pattern holds for the gravity sector.

$$h_a{}^b \ , \ h_{a, b_1 \ldots b_8} \ , \ \ldots \ , \ h_{a, b_1 \ldots b_8, c_1 \ldots c_9} , \ldots$$

which obey an infinite set of E_{11} related duality equations.

$$\omega_{a, b_1 b_2} = \epsilon_{b_1 b_2}{}^{c_1 \ldots c_9} (\partial_{c_1} A_{c_2 \ldots c_9} + \ldots)$$

etc.

$\rightsquigarrow$ Einstein equation $R_{ab} + \ldots = 0$ and

$$\partial^{[d} \partial_{[d} h_{a_1 \ldots a_8],}{}^{c]} + \ldots = 0$$