Massive and Partially Massless (PM) gravitons on curved space-times

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CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

FP7/2007-2013
« NIRG » project no. 307934

1. A short review on Massive Gravity

2. Massive graviton on arbitrary spacetimes.

L. Bernard, C.D., M. von Strauss + A. Schmidt-May (2015-2016, PR**D**, JCAP)

3. PM graviton on non Einstein spacetimes.

L. Bernard, C.D., K. Hinterbichler and M. von Strauss arXiv:1703.02538 (PR**D**)

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3. Erratum necessary for some technicalities in

L. Bernard, C.D., K. Hinterbichler and M. von Strauss
arXiv:1703.02538 (PR**D**)

(thanks to Charles Mazuet)

1. A short review on Massive Gravity

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One way to modify gravity at « large distances »
… and get rid of dark energy (or dark matter) ?

$$H^2 = \frac{8\pi G}{3}\rho$$

Changing the dynamics
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Dark matter
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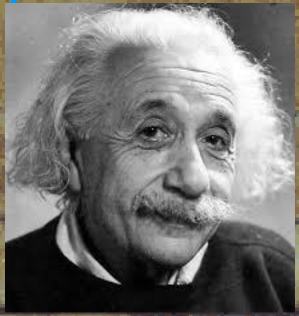

Historical example the success/failure of both approaches: Le Verrier and

- The discovery of Neptune
- The non discovery of Vulcan… but that of General Relativity

for this idea to work…

I.e. to « replace » the cosmological constant by a non vanishing graviton mass…

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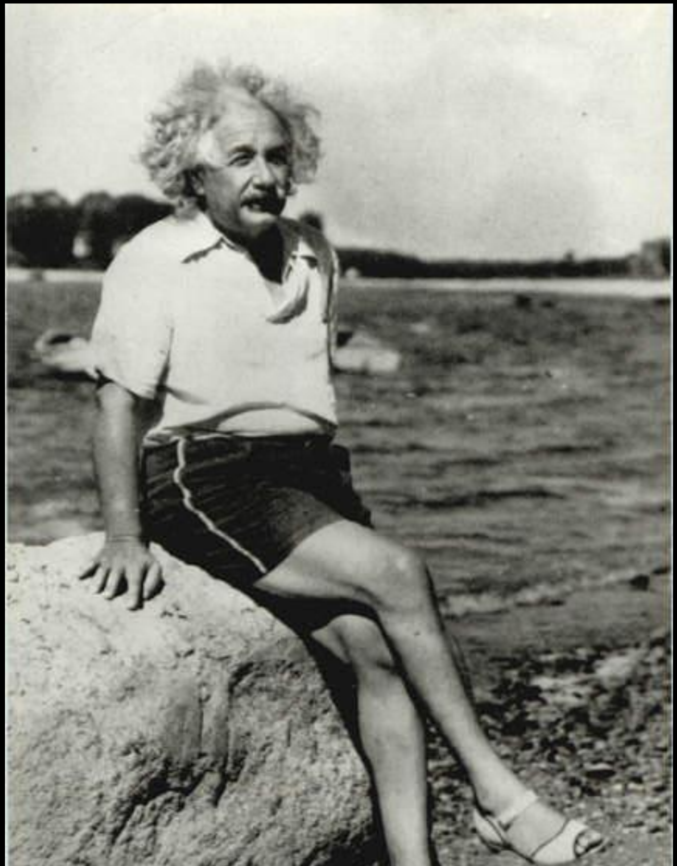
➡ One obviously needs a very light graviton (of Compton length of order of the size of the Universe)

for this idea to work…

I.e. to « replace » the cosmological constant by a non vanishing graviton mass…

NB: It seems one of the Einstein's motivations to introduce the cosmological constant was to try to « give a mass to the graviton »

(see « Einstein's mistake and the cosmological constant »
by A. Harvey and E. Schucking,
Am. J. of Phys. Vol. 68, Issue 8 (2000))



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1.1. Why massive gravity ?

➡ One way to modify gravity at « large distances »
… and get rid of dark energy (or dark matter) ?

$$H^2 = \frac{8\pi G}{3}\rho$$

Changing the dynamics of gravity ?

Dark matter dark energy ?

➡ Theoretical challenge to give a mass to the graviton

(here we will rather stay on this « abstract » side rather than discussing real world applications)

1. A short review on Massive Gravity

1.2. Fierz-Pauli theory on Einstein space-times

Consider an Einstein space-time obeying $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$

Fierz-Pauli theory (1939)

is the (<u>only correct</u>) theory of a massive graviton $h_{\mu\nu}$ which propagates on this space-time

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Consider an Einstein space-time obeying $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$

Fierz-Pauli theory is defined by

Fierz-Pauli (1939), Deser Nepomechie (1984), Higuchi (1987), Bengtsson (1995), Porrati (2001)

Field equations $E_{\mu\nu} \simeq 0$

on shell

with

$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) + \frac{m^2}{2} \left(h_{\mu\nu} - g_{\mu\nu} h \right)$$

Kinetic operator

Cosmological constant

Mass term

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Comes from expanding the Einstein-Hilbert action $\int d^4x \sqrt{-g} \left(R - 2\Lambda \right)$

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Kinetic operator

Cosmological constant

Mass term

Comes from expanding the

Einstein-H

Analogous to Proca equations for a massive photon :

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

$$E_{\mu\nu} \equiv \underbrace{\mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda\left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h\right)} + \frac{m^2}{2}\left(h_{\mu\nu} - g_{\mu\nu} h\right)$$

Kinetic operator

Cosmological constant

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Comes from expanding the

Einstein-Hilbert action $\int d^4x \sqrt{-g}\,(R - 2\Lambda)$

Interest: Cosmology ?
The mass term leads to « self acceleration » (C.D., Dvali, Gabadadze 2001)

DOF (= « polarizations ») counting

The Fierz Pauli theory for a massive graviton of mass m propagates

- 2 DOF if $m = 0$

Massless graviton (of GR)

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Generic massive graviton

- 5 DOF if $m \neq 0$ and $m^2 \neq 2\Lambda/3$

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the Proca « photon »
has 3 polarizations

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- 4 DOF if $m^2 = 2\,\Lambda/3$

Partially Massless graviton

How to count DOF ?

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$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) + \frac{m^2}{2} \left(h_{\mu\nu} - g_{\mu\nu} h \right)$$

Kinetic operator

Cosmological constant

Mass term

Comes from expanding the

Einstein-Hilbert action $\int d^4x \sqrt{-g}\,(R - 2\Lambda)$

This implies the (Bianchi) offshell identities

$$\nabla^{\mu} \left[\mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) \right] = 0$$

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Kinetic operator

Cosmological constant

Mass term

Come

Eins

Analogous to $\partial_\nu \partial_\mu F^{\mu\nu} = 0$

In Maxwell and Proca theories

This implies the (Bi) offshell identities

$$\nabla^\mu \left[\mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) \right] = 0$$

Results in an the off-shell identity

$$\nabla^{\mu} E_{\mu\nu} = \frac{m^2}{2} \left(\nabla^{\mu} h_{\mu\nu} - g^{\rho\sigma} \nabla_{\nu} h_{\rho\sigma} \right)$$

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And the on-shell relation

$$\nabla^{\mu} h_{\mu\nu} - \nabla_{\nu} h \simeq 0$$

4 vector
constraints

Kills 4 out of 10 DOF of $h_{\mu\nu}$

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$$\nabla^{\mu}$$

Analogous to the constraint

$$m^2\partial_\nu A^\nu \simeq 0$$

of Proca theory

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Taking an extra derivative of the field equation operator yields (off shell)

$$\nabla^\mu \nabla^\nu E_{\mu\nu} = \frac{m^2}{2} \left(\nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^2 h \right)$$

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While tracing it with the metric gives

$$g^{\mu\nu} E_{\mu\nu} = \nabla^2 h - \nabla^\mu \nabla^\nu h_{\mu\nu} + \left(\Lambda - \frac{3m^2}{2} \right) h$$

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Hence we have the identity

$$2\nabla^\mu \nabla^\nu E_{\mu\nu} + m^2 g^{\mu\nu} E_{\mu\nu} = \frac{m^2}{2}\left(2\Lambda - 3m^2\right) h$$

Yielding on shell $\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$

$$\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$$

Generically: yields $h \simeq 0$

i.e. a "scalar constraint" $\mathcal{C} \equiv h \simeq 0$

reducing from 6 to 5 the

number of propagating DOF

$$\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$$

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As is $2\nabla^\mu\nabla^\nu E_{\mu\nu} + m^2 g^{\mu\nu} E_{\mu\nu}$

Shows the existence of a gauge symmetry

$$\Delta h_{\mu\nu} = \left(\nabla_\mu\nabla_\nu + \frac{m^2}{2} g_{\mu\nu}\right)\xi(x) = \left(\nabla_\mu\nabla_\nu + \frac{\Lambda}{3} g_{\mu\nu}\right)\xi(x)$$

Hence, if $2\Lambda = 3m^2$

one has 6 - 2 = 4 DOF

The massive graviton is said to be "Partially massless" (PM)

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1.3. Non linear completions of Fierz-Pauli theory

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Stays sending the mass of the graviton smoothly to zero (**v**an **D**am, **V**eltman, **Z**akharov, Iwasaki 1970)

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A way out relying on non linearities was suggested by Arkady in 1972 (« Vainshtein mechanism »)

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This was overcome in 2010-2011 by de Rham, Gababadze, Tolley (« dRGT theory »)

dRGT theory

de Rahm, Gabadadze; de Rham, Gababadze, Tolley; Hassan, Rosen 2010, 2011

$$S_{g,m} = M_g^2 \int \mathrm{d}^4x \sqrt{|g|}\left[R(g) - 2m^2 V\left(S;\beta_n\right)\right]$$

Where $V(S;\beta_n) = \sum_{n=0}^{3} \beta_n e_n(S)$

with

$$\left\{ \begin{array}{l} \left\{ \begin{array}{rcl} e_1\left(\mathbf{S}\right) & = & tr\ \mathbf{S} \\ e_2\left(\mathbf{S}\right) & = & \frac{1}{2}\left(\left(tr\ \mathbf{S}\right)^2 - tr\ \mathbf{S}^2\right) \\ e_3\left(\mathbf{S}\right) & = & \frac{1}{6}\left(\left(tr\ \mathbf{S}\right)^3 - 3\ tr\ \mathbf{S}\ tr\ \mathbf{S}^2 + 2\ tr\ \mathbf{S}^3\right) \end{array} \right. \\ \left\{ S^{\mu}{}_{\sigma} S^{\sigma}{}_{\nu} = g^{\mu\sigma} f_{\sigma\nu} = \mathfrak{F}^{\mu}{}_{\nu} \right. \end{array} \right.$$

Has been shown to propagate 5 (or less d.o.f.) in a fully non linear way …

… evading Boulware-Deser no-go « theorem »

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Open question with some attempts and no-go results (de Rham, Hinterbichler, Rosen, Tolley; Deser, Waldron…)

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A related question: can a PM graviton exist on a non Einstein space-time ?

Addressed here…

First, we need to introduce the theory of a massive graviton on arbitrary backgrounds

2. Consistent massive graviton on arbitrary backgrounds

$$S^{(2)} = -\frac{1}{2} M_g^2 \int \mathrm{d}^4 x \sqrt{|g|} h_{\mu\nu} \left(\tilde{\mathcal{E}}^{\mu\nu\rho\sigma} + m^2 \mathcal{M}^{\mu\nu\rho\sigma} \right) h_{\rho\sigma}$$

Einstein-Hilbert
kinetic operator

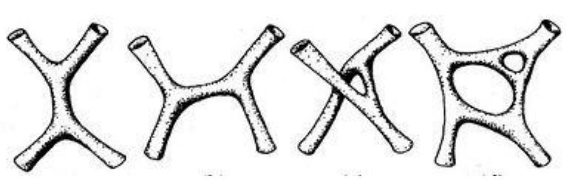
Mass term

The theory has been obtained in

L.Bernard, CD, M. von Strauss
1410.8302 + 1504.04382
+ 1512.03620 (with A. Schmidt-May)
(see also C. Mazuet, M. Volkov 2015)

out of the dRGT theory

Can be compared with Buchbinder, Gitman, Krykhtin, Pershin (2000)



Our massive graviton theory is defined by

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1. A symmetric tensor $S_{\mu\nu}$ obtained from the background curvature solving

$$R^{\mu}{}_{\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2} e_1 \beta_1 \right) \delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu} \right]$$

with β_0, β_1 and β_2 dimensionless parameters and *m* the graviton mass, e_i the symmetric polynomials

$$\begin{cases} e_0 = 1\,, \\ e_1 = S^{\rho}{}_{\rho}\,, \\ e_2 = \dfrac{1}{2}\left(S^{\rho}{}_{\rho} S^{\nu}{}_{\nu} - S^{\rho}{}_{\nu} S^{\nu}{}_{\rho}\right)\,, \\ e_3 = \dfrac{1}{6}\left(S^{\rho}{}_{\rho} S^{\nu}{}_{\nu} S^{\mu}{}_{\mu} - 3 S^{\mu}{}_{\mu} S^{\rho}{}_{\nu} S^{\nu}{}_{\rho} + 2 S^{\rho}{}_{\nu} S^{\nu}{}_{\mu} S^{\mu}{}_{\rho}\right)\,, \\ e_4 = \det(S)\,. \end{cases}$$

2. The following (linear) field equations

$$
\begin{aligned}
E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{m^2}{2} \Big[& 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right) \left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right) \\
& - \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma} \\
& - \left(\beta_1 + \beta_2 e_1\right) \left(g_{\mu\rho} \delta S^{\rho}{}_{\nu} + g_{\nu\rho} \delta S^{\rho}{}_{\mu}\right) \Big] \simeq 0 \,,
\end{aligned}
$$

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$$- \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma}$$
$$- \left(\beta_1 + \beta_2 e_1\right)\left(g_{\mu\rho}\delta S^\rho{}_\nu + g_{\nu\rho}\delta S^\rho{}_\mu\right)\Big] \simeq 0\,,$$

Linearized Einstein operator

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \equiv -\frac{1}{2}\Big[\delta^\rho_\mu \delta^\sigma_\nu \nabla^2 + g^{\rho\sigma}\nabla_\mu \nabla_\nu - \delta^\rho_\mu \nabla^\sigma \nabla_\nu - \delta^\rho_\nu \nabla^\sigma \nabla_\mu - g_{\mu\nu} g^{\rho\sigma} \nabla^2 + g_{\mu\nu} \nabla^\rho \nabla^\sigma$$
$$+ \delta^\rho_\mu \delta^\sigma_\nu R - g_{\mu\nu} R^{\rho\sigma}\Big] h_{\rho\sigma}\,,$$

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\begin{aligned}
E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{m^2}{2}\Big[& 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right)\left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right) \\
& - \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma} \\
& - \left(\beta_1 + \beta_2 e_1\right)\left(g_{\mu\rho} \boxed{\delta S^{\rho}{}_{\nu}} + g_{\nu\rho} \delta S^{\rho}{}_{\mu}\right)\Big] \simeq 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\lambda}{}_{\mu} = \frac{1}{2} g^{\nu\lambda} \Big[& e_4\, c_1 \left(\delta^{\rho}_{\nu} \delta^{\sigma}_{\mu} + \delta^{\sigma}_{\nu} \delta^{\rho}_{\mu} - g_{\mu\nu} g^{\rho\sigma}\right) + e_4\, c_2 \left(S^{\rho}_{\nu} \delta^{\sigma}_{\mu} + S^{\sigma}_{\nu} \delta^{\rho}_{\mu} - S_{\mu\nu} g^{\rho\sigma} - g_{\mu\nu} S^{\rho\sigma}\right) \\
& - e_3\, c_1 \left(\delta^{\rho}_{\nu} S^{\sigma}_{\mu} + \delta^{\sigma}_{\nu} S^{\rho}_{\mu}\right) + \left(e_2\, c_1 - e_4\, c_3 + e_3\, c_2\right)\, S_{\mu\nu} S^{\rho\sigma} \\
& + e_4\, c_3 \left[\delta^{\sigma}_{\mu} [S^2]^{\rho}_{\nu} + \delta^{\rho}_{\mu} [S^2]^{\sigma}_{\nu} - g^{\rho\sigma} [S^2]_{\mu\nu} + \delta^{\rho}_{\nu} [S^2]^{\sigma}_{\mu} + \delta^{\sigma}_{\nu} [S^2]^{\rho}_{\mu} - g_{\mu\nu} [S^2]^{\rho\sigma}\right] \\
& - e_3\, c_2 \left(S^{\rho}_{\nu} S^{\sigma}_{\mu} + S^{\sigma}_{\nu} S^{\rho}_{\mu}\right) - e_3\, c_3 \left(S^{\sigma}_{\mu} [S^2]^{\rho}_{\nu} + S^{\rho}_{\mu} [S^2]^{\sigma}_{\nu} + S^{\rho}_{\nu} [S^2]^{\sigma}_{\mu} + S^{\sigma}_{\nu} [S^2]^{\rho}_{\mu}\right) \\
& + \left(e_3\, c_3 - e_1\, c_1\right)\, \left(S^{\rho\sigma} [S^2]_{\mu\nu} + S_{\mu\nu} [S^2]^{\rho\sigma}\right) - \left(c_1 - e_2\, c_3\right)\, \left([S^2]^{\rho}_{\nu} [S^2]^{\sigma}_{\mu} + [S^2]^{\sigma}_{\nu} [S^2]^{\rho}_{\mu}\right) \\
& + c_4\, [S^2]_{\mu\nu} [S^2]^{\rho\sigma} + c_1 \left([S^3]_{\mu\nu} S^{\rho\sigma} + S_{\mu\nu} [S^3]^{\rho\sigma}\right) + c_2 \left([S^3]_{\mu\nu} [S^2]^{\rho\sigma} + [S^2]_{\mu\nu} [S^3]^{\rho\sigma}\right) \\
& + c_3\, [S^3]_{\mu\nu} [S^3]^{\rho\sigma} \Big]\, h_{\rho\sigma}\,,
\end{aligned}
$$

DOF counting ?

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The linearized Bianchi identity $$\nabla^{\mu}\mathcal{G}_{\mu\nu} = 0$$

Yields 4 vector constaints reducing from 10 to 6 the number of propagating DOF

DOF counting ?

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The linearized Bianchi identity $$\nabla^\mu \mathcal{G}_{\mu\nu} = 0$$

Yields 4 vector constaints reducing from 10 to 6 the number of propagating DOF

A scalar constraint $\mathcal{C} \simeq 0$ reduces to 5 the number of DOF

$$\mathcal{C} \equiv (S^{-1})^\nu{}_\rho \nabla^\rho \nabla^\mu E_{\mu\nu} + \frac{m^2\beta_1}{2} g^{\mu\nu} E_{\mu\nu} + m^2 \beta_2 S^{\mu\nu} E_{\mu\nu} \simeq 0$$

3. PM graviton on non Einstein space-times ?

We look for (non Einstein) space-times where the scalar constraint

$$\mathcal{C} \equiv (S^{-1})^{\nu}{}_{\rho} \nabla^{\rho} \nabla^{\mu} E_{\mu\nu} + \frac{m^2 \beta_1}{2} g^{\mu\nu} E_{\mu\nu} + m^2 \beta_2 S^{\mu\nu} E_{\mu\nu} \simeq 0$$

Identically vanishes

[Yielding the gauge symmetry $h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu}$ with

$$\Delta h_{\mu\nu} = \left[(S^{-1})_{\mu}{}^{\rho} \nabla_{\rho} \nabla_{\nu} + (S^{-1})_{\nu}{}^{\rho} \nabla_{\rho} \nabla_{\mu} + m^2 \beta_1 g_{\mu\nu} + 2 m^2 \beta_2 S_{\mu\nu} \right] \xi(x)$$
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We need to look in detail at the structure of the constraint

The scalar constraint reads

$$\mathcal{C} = m^2\left[\left(A^{\beta\lambda} + \tilde{A}^{\beta\lambda}\right)\tilde{h}_{\beta\lambda} + B_{\rho}^{\beta\lambda}\,\nabla^{\rho}\tilde{h}_{\beta\lambda}\right]$$

With $h_{\mu\nu} = \left(S^{\lambda}{}_{\mu}\delta^{\beta}_{\nu} + S^{\lambda}{}_{\nu}\delta^{\beta}_{\mu}\right)\tilde{h}_{\beta\lambda}$

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With $h_{\mu\nu} = \left(S^{\lambda}{}_{\mu} \delta^{\beta}_{\nu} + S^{\lambda}{}_{\nu} \delta^{\beta}_{\mu} \right) \tilde{h}_{\beta\lambda}$

$$\begin{aligned} A^{\beta\lambda} \equiv \; & m^2 \, S^{\beta}{}_{\rho} \Big[\left(\beta_0\beta_1 + \beta_0\beta_2 e_1 + \frac{1}{2}\beta_1^2 e_1 + \frac{1}{2}\beta_1\beta_2 e_1^2 \right) g^{\rho\lambda} \\ & + \left(-2\beta_0\beta_2 - \frac{1}{2}\beta_1^2 - \beta_1\beta_2 e_1 - 2\beta_2^2 e_2 + \beta_2^2 e_1^2 \right) S^{\rho\lambda} \\ & - \left(\beta_1\beta_2 + \beta_2^2 e_1 \right) [S^2]^{\rho\lambda} \Big] \end{aligned}$$

The scalar constraint reads

$$\mathcal{C} = m^2 \Big[\Big(A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \Big) \tilde{h}_{\beta\lambda} + B_{\rho}^{\beta\lambda} \nabla^{\rho} \tilde{h}_{\beta\lambda} \Big]$$

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$$\begin{aligned} A^{\beta\lambda} \equiv\ & m^2\, S^{\beta}{}_{\rho} \Big[\Big(\beta_0\beta_1 + \beta_0\beta_2 e_1 + \frac{1}{2}\beta_1^2 e_1 + \frac{1}{2}\beta_1\beta_2 e_1^2 \Big) g^{\rho\lambda} \\ & + \Big(-2\beta_0\beta_2 - \frac{1}{2}\beta_1^2 - \beta_1\beta_2 e_1 - 2\beta_2^2 e_2 + \beta_2^2 e_1^2 \Big) S^{\rho\lambda} \\ & - \left(\beta_1\beta_2 + \beta_2^2 e_1 \right) [S^2]^{\rho\lambda} \Big] \end{aligned}$$

Missing terms in v1

The scalar constraint reads

$$\mathcal{C} = m^2 \Big[\Big(A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \Big) \tilde{h}_{\beta\lambda} + B_\rho^{\beta\lambda} \nabla^\rho \tilde{h}_{\beta\lambda} \Big]$$

With $h_{\mu\nu} = \Big(S^\lambda{}_\mu \delta^\beta_\nu + S^\lambda{}_\nu \delta^\beta_\mu \Big) \tilde{h}_{\beta\lambda}$

$$\begin{aligned}
\tilde{A}^{\beta\lambda} \equiv \frac{1}{2} (\beta_1 + \beta_2 e_1) [S^{-1}]^\nu_\gamma \Big[& -\nabla^\gamma S^{\rho\lambda} \nabla_\nu S^\beta_\rho + \nabla^\gamma S^\beta_\rho \nabla^\lambda S^\rho_\nu + \nabla^\gamma S^\rho_\nu \nabla^\lambda S^\beta_\rho - \nabla^\gamma S_{\rho\nu} \nabla^\rho S^{\beta\lambda} \\
& - S^{\rho\lambda} \nabla^\gamma \nabla_\nu S^\beta_\rho + S^\beta_\rho \nabla^\gamma \nabla^\lambda S^\rho_\nu \Big] + \beta_2 [S^{-1}]^\nu{}_\gamma \Big[S^\beta_\rho \nabla^\lambda S^\rho_\nu \nabla^\gamma e_1 - S^\beta_\rho \nabla_\nu S^{\rho\lambda} \nabla^\gamma e_1 \\
& + S^\lambda_\rho \nabla^\gamma S^\beta_\mu \nabla_\nu S^{\rho\mu} + S^\beta_\mu \nabla^\gamma S^\lambda_\rho \nabla_\nu S^{\rho\mu} + S^\lambda_\mu \nabla^\gamma S^{\mu\rho} \nabla_\nu S^\beta_\rho + S^{\mu\rho} \nabla^\gamma S^\lambda_\mu \nabla_\nu S^\beta_\rho \\
& - 2 S^\beta_\mu \nabla^\gamma S^{\mu\lambda} \nabla_\nu e_1 - S^\rho_\mu \nabla^\gamma S^\mu_\nu \nabla^\beta S^\lambda_\rho - S^\beta_\mu \nabla^\gamma S^\mu_\rho \nabla^\lambda S^\rho_\nu - S^\mu_\rho \nabla^\gamma S^\beta_\mu \nabla^\lambda S^\rho_\nu \\
& - S^\beta_\rho \nabla^\gamma S^\mu_\nu \nabla^\lambda S^\rho_\mu + S^\beta_\mu \nabla^\gamma S^\mu_\nu \nabla^\lambda e_1 + S^\mu_\rho \nabla^\gamma S_{\mu\nu} \nabla^\rho S^{\beta\lambda} - S^\lambda_\mu \nabla^\gamma S^\mu_\nu \nabla^\rho S^\beta_\rho \\
& - S^\lambda_\mu \nabla^\gamma S^\beta_\rho \nabla^\mu S^\rho_\nu - S^\beta_\rho \nabla^\gamma S^\lambda_\mu \nabla^\mu S^\rho_\nu - S^\lambda_\mu \nabla^\gamma S^\rho_\nu \nabla^\mu S^\beta_\rho + 2 S^\beta_\mu \nabla^\gamma S^{\mu\lambda} \nabla^\rho S_{\rho\nu} \\
& + 2 S^\beta_\rho \nabla^\gamma S_{\mu\nu} \nabla^\mu S^{\rho\lambda} + S^\lambda_\rho S^\beta_\mu \nabla^\gamma \nabla_\nu S^{\rho\mu} + [S^2]^{\lambda\rho} \nabla^\gamma \nabla_\nu S^\beta_\rho - [S^2]^{\beta\lambda} \nabla^\gamma \nabla_\nu e_1 \\
& - [S^2]^\beta_\rho \nabla^\gamma \nabla^\lambda S^\rho_\nu - S^\lambda_\mu S^\beta_\rho \nabla^\gamma \nabla^\mu S^\rho_\nu + [S^2]^{\beta\lambda} \nabla^\gamma \nabla^\rho S_{\rho\nu} \Big] + \beta_2 \Big[+ \nabla^\beta S^\lambda_\gamma \nabla^\gamma e_1 \\
& - \nabla_\gamma S^{\beta\lambda} \nabla^\gamma e_1 - \nabla^\mu S^\rho_\mu \nabla^\beta S^\lambda_\rho - \nabla^\mu S^\beta_\rho \nabla^\lambda S^\rho_\mu + \nabla^\mu S^\beta_\mu \nabla^\lambda e_1 + \nabla_\mu S^\mu_\rho \nabla^\rho S^{\beta\lambda} \\
& - \nabla^\mu S^\lambda_\mu \nabla^\rho S^\beta_\rho - \nabla^\rho S^\lambda_\mu \nabla^\mu S^\beta_\rho + 2 \nabla_\mu S^\beta_\rho \nabla^\mu S^{\rho\lambda} - S^\beta_\rho \nabla^\lambda \nabla^\mu S^\rho_\mu + S^\beta_\gamma \nabla^\gamma \nabla^\lambda e_1 \\
& - S^\lambda_\gamma \nabla^\gamma \nabla^\rho S^\beta_\rho + S^\beta_\rho \nabla^\gamma \nabla_\gamma S^{\rho\lambda} \Big] + (\beta \leftrightarrow \lambda) \,,
\end{aligned}$$

The scalar constraint reads

$$\mathcal{C} = m^2 \Big[\Big(A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \Big) \tilde{h}_{\beta\lambda} + B^{\beta\lambda}_{\rho} \nabla^{\rho} \tilde{h}_{\beta\lambda} \Big]$$

With $h_{\mu\nu} = \Big(S^{\lambda}{}_{\mu} \delta^{\beta}_{\nu} + S^{\lambda}{}_{\nu} \delta^{\beta}_{\mu} \Big) \tilde{h}_{\beta\lambda}$

$$\begin{aligned}
B^{\beta\lambda}_{\rho} \equiv{}& \frac{1}{2} (\beta_1 + \beta_2 e_1) [S^{-1}]^{\nu}_{\gamma} \left[-S^{\sigma\lambda} \delta^{\gamma}_{\rho} \nabla_{\nu} S^{\beta}_{\sigma} + \delta^{\gamma}_{\rho} S^{\beta}_{\sigma} \nabla^{\lambda} S^{\sigma}_{\nu} + \delta^{\lambda}_{\rho} S^{\beta}_{\sigma} \nabla^{\gamma} S^{\sigma}_{\nu} - S^{\beta\lambda} \nabla^{\gamma} S_{\nu\rho} \right] \\
&+ \beta_2 [S^{-1}]^{\nu}{}_{\gamma} \Big[\delta^{\gamma}_{\rho} S^{\lambda}_{\delta} S^{\beta}_{\mu} \nabla_{\nu} S^{\delta\mu} + \delta^{\gamma}_{\rho} [S^2]^{\lambda\mu} \nabla_{\nu} S^{\beta}_{\mu} - \delta^{\gamma}_{\rho} [S^2]^{\beta\lambda} \nabla_{\nu} e_1 - \delta^{\gamma}_{\rho} [S^2]^{\beta}_{\mu} \nabla^{\lambda} S^{\mu}_{\nu} \\
&- \delta^{\gamma}_{\rho} S^{\lambda}_{\mu} S^{\beta}_{\delta} \nabla^{\mu} S^{\delta}_{\nu} + \delta^{\gamma}_{\rho} [S^2]^{\beta\lambda} \nabla^{\mu} S_{\mu\nu} + S^{\beta\lambda} S^{\mu}_{\rho} \nabla^{\gamma} S_{\mu\nu} + [S^2]^{\beta\lambda} \nabla^{\gamma} S_{\rho\nu} - \delta^{\beta}_{\rho} [S^2]^{\lambda}_{\mu} \nabla^{\gamma} S^{\mu}_{\nu} \\
&- S^{\beta}_{\rho} S^{\lambda}_{\mu} \nabla^{\gamma} S^{\mu}_{\nu} \Big] + \beta_2 \Big[-S^{\beta}_{\delta} \nabla^{\lambda} S^{\delta}_{\rho} + S^{\beta}_{\rho} \nabla^{\lambda} e_1 - S^{\lambda}_{\mu} \nabla^{\mu} S^{\beta}_{\rho} + 2 S^{\beta}_{\delta} \nabla_{\rho} S^{\delta\lambda} + \delta^{\beta}_{\rho} S^{\lambda}_{\gamma} \nabla^{\gamma} e_1 \\
&- S^{\beta\lambda} \nabla_{\rho} e_1 + S^{\beta\lambda} \nabla_{\mu} S^{\mu}_{\rho} - \delta^{\beta}_{\rho} S^{\lambda}_{\delta} \nabla^{\mu} S^{\delta}_{\mu} - S^{\beta}_{\rho} \nabla^{\mu} S^{\lambda}_{\mu} \Big] + (\beta \leftrightarrow \lambda) \,.
\end{aligned}$$

To get a PM theory we need to look for space-times

where $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$ vanish identically.

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Most general solution ?

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where $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_\rho^{\beta\lambda}$ vanish identically.

Most general solution ?

Assume $\boxed{\nabla_\rho S_{\mu\nu} = 0}$

(i.e. $S_{\mu\nu}$ covariantly constant)

makes $\tilde{A}^{\beta\lambda}$ and $B_\rho^{\beta\lambda}$ vanish.

Space-times possessing a covariantly constant tensor $H_{\mu\nu}$ are severely restricted…

$$\nabla_\rho H_{\mu\nu} = 0$$

$$\underbrace{\nabla_{[\mu}\nabla_{\nu]} H_{\rho\lambda} = R_{\mu\nu\rho}{}^{\sigma} H_{\sigma\lambda} + R_{\mu\nu\lambda}{}^{\sigma} H_{\rho\sigma}}$$

$$\Downarrow$$

$$R_{\mu\nu\rho}{}^{\sigma} H_{\sigma\lambda} + R_{\mu\nu\lambda}{}^{\sigma} H_{\rho\sigma} = 0$$

Non trivial integrability conditions

Space-times possessing a covariantly constant tensor $H_{\mu\nu}$ are classified as (provided $H_{\mu\nu}$ is not proportional to the metric)

1. Spacetime is $2 \otimes 2$ decomposable

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{ab}(x^c)\mathrm{d}x^a \mathrm{d}x^b + g_{ij}(x^k)\mathrm{d}x^i \mathrm{d}x^j$$

and $H_{\mu\rho}H^\rho{}_\nu = H_{\mu\nu}$

2. The spacetime admits a covariantly constant vector N^μ and $H_{\mu\nu} = N_\mu N_\nu$

Note further that the definition

$$R^{\mu}{}_{\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2} e_1 \beta_1 \right) \delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu} \right]$$

Imposes that $\nabla_{\rho} S_{\mu\nu} = 0 \Longrightarrow \nabla_{\rho} R_{\mu\nu} = 0$

Hence the space-time must be "Ricci Symmetric"

…i.e. the covariantly constant tensor is the Ricci tensor…

The spacetimes of interest for us here will all have

$(R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu$ with r_1 and r_2 constant

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\begin{cases} \left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2} e_1 + \frac{1}{2}\beta_1\beta_2 e_1^2\right) + S^\beta_\lambda\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2} - \beta_1\beta_2 e_1\right) - (S^2)^\beta_\lambda(\beta_1\beta_2 + e_1\beta_2^2) = 0 \\ R^\mu{}_\nu = m^2\left[\left(\beta_0 + \frac{1}{2}e_1\beta_1\right)\delta^\mu_\nu + (\beta_1 + \beta_2 e_1)\, S^\mu{}_\nu - \beta_2 (S^2)^\mu{}_\nu\right] \end{cases}$$

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In order to get a vanishing $\mathcal{C} = m^2 A^{\beta\lambda}\tilde{h}_{\beta\lambda}$

The spacetimes of interest for us here will all have

$$(R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu \quad \text{with } r_1 \text{ and } r_2 \text{ constant}$$

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\left\{\begin{array}{l} \left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2}e_1 + \frac{1}{2}\beta_1\beta_2 e_1^2\right) + S^\beta_\lambda\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2} - \beta_1\beta_2 e_1\right) - (S^2)^\beta_\lambda\left(\beta_1\beta_2 + e_1\beta_2^2\right) = 0 \\ \boxed{R^\mu{}_\nu = m^2\left[\left(\beta_0 + \frac{1}{2}e_1\beta_1\right)\delta^\mu_\nu + \left(\beta_1 + \beta_2 e_1\right)S^\mu{}_\nu - \beta_2 (S^2)^\mu{}_\nu\right]} \end{array}\right.$$

From the definition of $S^\rho{}_\nu$

The spacetimes of interest for us here will all have

$$(R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu$$ with r_1 and r_2 constant

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\begin{cases} \left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2} e_1 + \frac{1}{2}\beta_1\beta_2 e_1^2\right) + S^\beta_\lambda \left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2} - \beta_1\beta_2 e_1\right) - (S^2)^\beta_\lambda \left(\beta_1\beta_2 + e_1\beta_2^2\right) = 0 \\ \boxed{R^\mu{}_\nu = m^2 \left[\left(\beta_0 + \frac{1}{2} e_1 \beta_1\right)\delta^\mu_\nu + (\beta_1 + \beta_2 e_1)\, S^\mu{}_\nu - \beta_2 (S^2)^\mu{}_\nu\right]} \end{cases}$$

NB: this implies that the Ricci tensor obeyes indeed the required relation

Explicit solutions (I)

with
$$\begin{cases} (R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu \\ \boxed{u \equiv \dfrac{\beta_0\beta_2}{\beta_1^2}} \qquad \boxed{v \equiv m^2\beta_0} \end{cases}$$

dimensionless

dimensionful

$$2\otimes 2 \begin{cases} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \dfrac{2\mathrm{d}x^0\mathrm{d}x^1}{(1+(R-E)x^0x^1/8)^2} - \dfrac{2\mathrm{d}x^2\mathrm{d}x^3}{(1-(R+E)x^2x^3/8)^2} \\ \text{type D}: \quad r_1 = \dfrac{R}{2}\,, \qquad r_2 = -\dfrac{1}{16}\left(R^2-E^2\right) \\ u = \dfrac{E-3R}{4(E-R)}, \quad \boxed{v = \dfrac{(E-3R)(E-R)}{8E}} \\ u = \dfrac{E+3R}{4(E+R)}, \quad \boxed{v = -\dfrac{(E+3R)(E+R)}{8E}} \end{cases}$$

Analogous to $2\Lambda = 3m^2$

For all these solutions one has

$$S^{\rho}{}_{\nu} = \frac{2\beta_1}{m^2 c} R^{\rho}{}_{\nu}$$

There also exist another set of Petrov type D solutions for which

$$S^{\mu}_{\nu} = -\frac{\beta_1}{\beta_2}\left(u - \frac{1}{2}\right)\delta^{\mu}_{\nu} + \frac{\beta_1}{\beta_2 R}(-3 + 4u)R^{\mu}_{\nu}$$

Explicit solutions (II)

with $(R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu$

$$1 \otimes 3 \left[\begin{array}{l} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \dfrac{\mathrm{d}x^2 + \mathrm{d}y^2 - \epsilon\, \mathrm{d}w^2}{\left[1 + R\,(x^2 + y^2 - \epsilon\, w^2)/24\right]^2} + \epsilon\, \mathrm{d}z^2 \\ \text{type O}: \quad r_1 = \dfrac{R}{3}\,, \qquad r_2 = 0 \end{array} \right.$$

No type O solutions

(Einstein static Universe is gone as a solution in our v2)

Explicit solutions (II)

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No type O solutions

(if one chooses the simplest kind of matrix square root solving for $S_\mu{}^\nu$)

(Einstein static Universe is gone as a solution in our v2)

Conclusions

PM exists on non Einstein spacetimes !

(in contrast with previous no-go claim by Deser, Joung, Waldron
In 1208.1307 [hep-th]…)

Solution for the vanishing
Of $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$
are not known in full generality !