

The $SL_q(2)$ extension of the standard model

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The idea that the elementary particles might have the symmetry of knots has had a long history. In any modern formulation of this idea, however, the knot must be quantized. The present review is a summary of a small set of papers that began as an attempt to correlate the properties of quantized knots with empirical properties of the elementary particles. As the ideas behind these papers have developed over a number of years, the model has evolved, and this review is intended to present the model in its current form. The original picture of an elementary fermion as a solitonic knot of field, described by the trefoil representation of $SU_q(2)$, has expanded into its present form in which a knotted field is complementary to a composite structure composed of three preons that in turn are described by the fundamental representation of $SL_q(2)$. Higher representations of $SL_q(2)$ are interpreted as describing composite particles composed of three or more preons bound by a knotted field. This preon model unexpectedly agrees in important detail with the Harari–Shupe model. There is an associated Lagrangian dynamics capable in principle of describing the interactions and masses of the particles generated by the model.

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1. Introduction

The possibility that the elementary particles are knotted has been suggested by many authors, going back as far as Kelvin, Maxwell and Tait.¹ Among the different field theoretic attempts to construct classical knots, a model related to the Skyrme soliton has been described by Faddeev and Niemi.² There are also the familiar knots of magnetic field; and since these are macroscopic expressions of the electroweak field, it is natural to extrapolate from macroscopic to microscopic knots of this same field. One expects that the conjectured microscopic knots would be quantized, and that they would be observed as solitonic in virtue of both their topological and quantum stability. It is then natural to ask if the elementary particles might also be knotted. If they are, one expects that the most elementary particles, namely the elementary fermions, are the most elementary knots, namely the trefoils. This possibility is also suggested by the fact that there are four quantum trefoils and four families of elementary fermions (charged leptons, neutrinos, and up and down quarks), and is supported by a unique one-to-one correspondence between the topological description of the four quantum trefoils and the electroweak quantum numbers of the four fermionic families. We have first attempted to determine the minimal restrictions on a model of the elementary particles in the context of electroweak interactions if the knotted soliton (quantum knot) is described only by its symmetry algebra $SL_q(2)$ independent of its field theoretic origin. The use of this symmetry algebra to define the quantum knot is similar to the use of the symmetry algebra of the rotation group to define the quantum spin. Before describing the symmetry algebra $SL_q(2)$ we shall describe an oriented classical knot by its topological invariants and by an invariant polynomial.

2. The Characterization of Oriented Knots

Three-dimensional knots are described in terms of their projections onto a two-dimensional plane where they appear as two-dimensional curves with 4-valent vertices. At each vertex (crossing) there is an overline and an underline. We shall be interested here in oriented knots. The crossing sign of the vertex is $+1$ or -1 depending on whether the direction of the overline is carried into the direction of the underline by a counterclockwise or clockwise rotation, respectively. The sum of all crossing signs is termed the writhe, w , a topological invariant. There is a second topological invariant, the rotation, r , the number of rotations of the tangent in going once around the knot. These are invariants of regular (two-dimensional) isotopy but not of ambient (three-dimensional) isotopy.

We may label an oriented knot by the number of crossings (N), the writhe (w) and rotation (r). The writhe and rotation are integers of opposite parity.

3. The Kauffman Algorithm for Associating a Polynomial with a Knot³

Denote the Kauffman polynomial associated with a knot, K , having n crossings, by $\langle K \rangle_n$. Let us represent $\langle K \rangle_n$ by the bracket

$$\langle K \rangle_n \sim \left\langle \begin{array}{c} \cdots \\ \times \\ \end{array} \right\rangle_n . \quad (3.1)$$

The interior of the bracket is intended to represent the projected knot when only one of the n crossings is explicitly shown. Let us also introduce the polynomials $\langle K_{\pm} \rangle_{n-1}$, associated with slightly altered diagrams in which the crossing lines are reconnected, as follows:

$$\langle K_- \rangle_{n-1} \sim \left\langle \begin{array}{c} \cdots \\ \succ \\ \end{array} \right\rangle_{n-1} \quad \text{and} \quad \langle K_+ \rangle_{n-1} \sim \left\langle \begin{array}{c} \cdots \\) \\ \end{array} \right\rangle_{n-1} . \quad (3.2)$$

Then one may define a Laurent polynomial in the parameter q by the following recursive rules:

$$\langle K \rangle_n = i[q^{-1/2}\langle K_- \rangle_{n-1} - q^{1/2}\langle K_+ \rangle_{n-1}] , \quad (\text{I})$$

$$\langle OK \rangle = (q + q^{-1})\langle K \rangle , \quad (\text{II})$$

$$\langle O \rangle = q + q^{-1} , \quad (\text{III})$$

where O is any closed loop generated by (I).

The Kauffman rules may be written entirely in terms of the Pauli matrices σ_{\pm} and the following matrix:

$$\varepsilon_q = \begin{pmatrix} 0 & q_1^{1/2} \\ -q^{1/2} & 0 \end{pmatrix} , \quad q_1 \equiv q^{-1} . \quad (3.3)$$

These rules then read as follows:

$$\langle K \rangle_n = \text{Tr } \varepsilon_q [\sigma_- \langle K_- \rangle_{n-1} + \sigma_+ \langle K_+ \rangle_{n-1}], \tag{I'}$$

$$\langle OK \rangle = \text{Tr } \varepsilon_q^t \varepsilon_q \langle K \rangle, \tag{II'}$$

$$\langle O \rangle = \text{Tr } \varepsilon_q^t \varepsilon_q, \tag{III'}$$

where t means transpose and

$$\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2).$$

Here the σ are the Pauli matrices.

One may obtain an invariant of ambient isotopy by forming^{3,4}

$$f_K(A) = (-A^3)^{-w(K)} \langle K \rangle, \tag{3.4}$$

where $w(K)$ is the writhe of K and

$$A = i \text{Tr } \varepsilon_q \sigma_- . \tag{3.5}$$

The Jones polynomial is

$$V_K(t) = f_K(t^{-1/4}). \tag{3.6}$$

The Kauffman and Jones polynomials are topological invariants. They are invariants of regular and ambient isotopy, respectively.

4. The Knot Algebra⁴⁻⁶

The description of the knot by (I'), (II'), (III') is invariant under the transformations

$$\varepsilon'_q = \text{T} \varepsilon_q \text{T}^t = \text{T}^t \varepsilon_q \text{T}, \tag{4.1a}$$

$$\sigma' = \sigma, \tag{4.1b}$$

where

$$\text{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.2}$$

and the matrix elements of T satisfy the following algebra:

$ab = qba, \quad bd = qdb, \quad ad - qbc = 1, \quad bc = cb,$	(A)
$ac = qca, \quad cd = qdc, \quad da - q_1cb = 1, \quad q_1 \equiv q^{-1}.$	

Then

$$\text{T} \varepsilon_q \text{T}^t = \text{T}^t \varepsilon_q \text{T} = \varepsilon_q \tag{4.3}$$

or by (4.1a)

$$\varepsilon'_q = \varepsilon_q. \tag{4.4}$$

Therefore the Kauffman algorithm as expressed in terms of ε_q is invariant under (4.1). We shall refer to (A) as the knot algebra. The matrix T, as defined by (4.2) and (A), is a two-dimensional representation of $SL_q(2)$.

We shall also introduce the unitary algebra $SU_q(2)$ obtained by setting

$$d = \bar{a}, \quad c = -q_1 \bar{b}. \tag{4.5}$$

Then (A) reduces to

$$\begin{aligned} ab &= qba, & a\bar{a} + b\bar{b} &= 1, \\ \bar{a}\bar{b} &= \bar{b}b, \\ \bar{a}\bar{b} &= q\bar{b}a, & \bar{a}a + q_1^2 \bar{b}b &= 1, \end{aligned} \tag{A'}$$

For the physical applications we require the higher-dimensional representations of $SL_q(2)$ and $SU_q(2)$.

5. Higher-Dimensional Representations of $SL_q(2)$ and $SU_q(2)$

To compute the higher-dimensional representations one needs the q -binomial theorem.⁷ This may be written in either of the following two ways:

$$(A + B)^n = \sum \left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_q B^s A^{n-s} \tag{5.1a}$$

or as

$$(A + B)^n = \sum \left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_{q_1} A^s B^{n-s}, \tag{5.1b}$$

where

$$AB = qBA \quad \text{and} \quad q_1 = q^{-1}. \tag{5.2}$$

Here

$$\left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_q = \frac{\langle n \rangle_{q!}}{\langle n-s \rangle_{q!} \langle s \rangle_{q!}} \quad \text{with} \quad \langle n \rangle_q = \frac{q^n - 1}{q - 1}. \tag{5.3}$$

We shall use this theorem to compute the $SL_q(2)$ transformations on the following class of monomials:

$$\psi_m^j = N_m^j x_1^{n+} x_2^{n-}, \quad -j \leq m \leq j, \tag{5.4a}$$

where

$$[x_1, x_2] = 0, \tag{5.4b}$$

$$n_{\pm} = j \pm m, \tag{5.4c}$$

$$N_m^j = \frac{1}{[\langle n_+ \rangle_{q_1}! \langle n_- \rangle_{q_1}!]^{1/2}}, \tag{5.4d}$$

when $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is transformed according to the two-dimensional representations of $SL_q(2)$ as follows:

$$x'_1 = ax_1 + bx_2, \tag{5.5}$$

$$x'_2 = cx_1 + dx_2, \tag{5.6}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the two-dimensional representation T of $SL_q(2)$ introduced in (4.2) and (A). Then

$$\psi_m^{j'} = N_m^j (ax_1 + bx_2)^{n_+} (cx_1 + dx_2)^{n_-}. \tag{5.7}$$

We assume that (a, b, c, d) commute with (x_1, x_2) so that

$$(ax_1)(bx_2) = q(bx_2)(ax_1), \tag{5.8}$$

$$(cx_1)(dx_2) = q(dx_2)(cx_1). \tag{5.9}$$

By the q -binomial theorem,

$$\begin{aligned} \psi_m^{j'} &= N_m^j \sum_s^{n_+} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} (ax_1)^s (bx_2)^{n_+-s} \sum_t^{n_-} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} (cx_1)^t (dx_2)^{n_- - t} \\ &= N_m^j \sum_{s,t} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} x_1^{s+t} x_2^{n_++n_--s-t} a^s b^{n_+-s} c^t d^{n_- - t} \end{aligned} \tag{5.10}$$

$$= N_m^j \sum_{s,t} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} a^s b^{n_+-s} c^t d^{n_- - t} x_1^{n'_+} x_2^{n'_-}, \tag{5.11}$$

where

$$n'_+ = s + t, \tag{5.12}$$

$$n'_- = n_+ + n_- - s - t \tag{5.13}$$

and by (5.4c)

$$n'_+ + n'_- = n_+ + n_- = 2j. \tag{5.14}$$

Set

$$n'_\pm = j \pm m'. \tag{5.15}$$

We may rewrite (5.11) as

$$\begin{aligned} \psi_m^{j'} &= \sum_{s,t} \left(\frac{N_m^j}{N_{m'}^j} \right) \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} \\ &\quad \times \delta(s + t, n'_+) a^s b^{n_+-s} c^t d^{n_- - t} \left(N_{m'}^j x_1^{n'_+} x_2^{n'_-} \right) \end{aligned} \tag{5.16}$$

$$= \sum_{m'} D_{mm'}^j \psi_{m'}^j, \tag{5.17}$$

where

$$D_{mm'}^j = \frac{N_m^j}{N_{m'}^j} \sum_{s,t} \langle n_+ \rangle_s \langle n_- \rangle_t \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (5.18)$$

or

$$D_{mm'}^j = \left(\frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right)^{\frac{1}{2}} \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \langle n_+ \rangle_s \langle n_- \rangle_t \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t}, \quad (5.19)$$

where we write $\langle \rangle_1$ for $\langle \rangle_{q_1}$. The corresponding representations of $SUq(2)$ are obtained by setting

$$d = \bar{a}, \quad (5.20a)$$

$$c = -q_1 \bar{b}. \quad (5.20b)$$

Then

$$D_{mm'}^j = \left(\frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right)^{\frac{1}{2}} \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \langle n_+ \rangle_s \langle n_- \rangle_t \times \delta(s+t, n'_+) (-q_1)^t a^s b^{n_+-s} \bar{b}^t \bar{a}^{n_--t}. \quad (5.21)$$

For both $SLq(2)$ and $SUq(2)$ we have

$$\psi_m^j(x'_1, x'_2) = \sum D_{mm'}^j \psi_{m'}^j(x_1, x_2). \quad (5.22)$$

In obtaining these representations of $SLq(2)$ and $SUq(2)$ that operate on the monomial basis (5.4a) we are following a well known procedure for obtaining representations of $SU(2)$.⁸

Let us rewrite (5.19) by introducing the exponents (n_a, n_b, n_c, n_d) . Then

$$n_a = s, \quad (5.23)$$

$$n_c = t, \quad (5.24)$$

$$n_+ = n_a + n_b, \quad (5.25)$$

$$n_- = n_c + n_d \quad (5.26)$$

and

$$D_{mm'}^j(q | a, b, c, d) = \sum_{\substack{\delta(n_a+n_b, n_+) \\ \delta(n_c+n_d, n_-)}} A_{mm'}^j(q | n_a, n_c) \delta(n_a + n_c, n'_+) a^{n_a} b^{n_b} c^{n_c} d^{n_d}, \quad (5.27)$$

where

$$A_{mm'}^j(q | n_a, n_c) = \left[\frac{\langle n'_+ \rangle_1 \langle n'_- \rangle_1}{\langle n_+ \rangle_1 \langle n_- \rangle_1} \right]^{\frac{1}{2}} \frac{\langle n_+ \rangle_1!}{\langle n_a \rangle_1! \langle n_b \rangle_1!} \frac{\langle n_- \rangle_1!}{\langle n_c \rangle_1! \langle n_d \rangle_1!}. \quad (5.28)$$

The sum (5.27) is over the positive integers n_a, n_b, n_c, n_d , subject to the constraints as shown. The n_{\pm} and n'_{\pm} are given by (5.4c) and (5.15) respectively.

6. The Gauge Group of the SLq(2) and SUq(2) Algebras

By (5.27) the $2j + 1$ -dimensional representations of the SLq(2) have the following form

$$D^j_{mm'} = \sum_{\substack{\delta(n_a+n_b, n_+) \\ \delta(n_c+n_d, n_-)}} A^j_{mm'}(q | n_a, n_c) \delta(n_a + n_c, n'_+) a^{n_a} b^{n_b} c^{n_c} d^{n_d}, \quad (6.1)$$

where (a, b, c, d) satisfy the knot algebra (A) defined in Sec. 4 and

$$n_{\pm} = j \pm m, \quad (6.2)$$

$$n'_{\pm} = j \pm m', \quad (6.3)$$

$D^j_{mm'}$ is defined only up to the following gauge transformation on (a, b, c, d) that leaves the algebra (A) invariant:

$$\begin{aligned} a' &= e^{i\varphi_a} a, & b' &= e^{i\varphi_b} b, \\ d' &= e^{-i\varphi_a} d, & c' &= e^{-i\varphi_b} c. \end{aligned} \quad (G)$$

We shall also refer to (G) as $U_a(1) \times U_b(1)$. Under the gauge transformation (G), every term in $D^j_{mm'}$ transforms like

$$(a^{n_a} b^{n_b} c^{n_c} d^{n_d})' = e^{i\varphi_a(n_a - n_d)} e^{i\varphi_b(n_b - n_c)} (a^{n_a} b^{n_b} c^{n_c} d^{n_d}). \quad (6.4)$$

But by the δ -functions in (6.1)

$$\begin{aligned} n_a - n_d &= n_a - (n_- - n_c) = (n_a + n_c) - n_- \\ &= n'_+ - n_- = m' + m, \\ n_b - n_c &= n_+ - n_a - n_c = n_+ - n'_+ = m - m'. \end{aligned} \quad (6.5)$$

By (6.4) and (6.5) every term of $D^j_{mm'}$ transforms the same way and therefore the $D^j_{mm'}$ transforms under (G) as follows:

$$D^j_{mm'}(a, b, c, d) \rightarrow e^{i(m+m')\varphi_a} e^{i(m-m')\varphi_b} D^j_{mm'}(a, b, c, d) \quad (6.6a)$$

or

$$D^j_{mm'} \rightarrow e^{i(\varphi_a + \varphi_b)m} e^{i(\varphi_a - \varphi_b)m'} D^j_{mm'}. \quad (6.6b)$$

We denote the irreducible representations of SUq(2) by $D^j_{mn}(a, \bar{a}, b, \bar{b})$.

The gauge transformations on SUq(2), namely

$$a' = e^{i\varphi_a} a, \quad b' = e^{-i\varphi_b} b, \quad (6.7)$$

induce similar transformations on the $D^j_{mn}(a, \bar{a}, b, \bar{b})$.

7. Representation of an Oriented Knot

The oriented 2d-projection of a knot has three coordinates, namely (N, w, r) the number of crossings N , the writhe w and the rotation r . We may make a coordinate transformation to (j, m, m') , the indices that label the irreducible representations $D_{mm'}^j$ of $SLq(2)$, the symmetry algebra of the knot, by setting

$$j = \frac{N}{2}, \quad m = \frac{w}{2}, \quad m' = \frac{(r + o)}{2}, \tag{7.1}$$

where o is an odd integer.

This linear transformation allows half-integer representations of $SLq(2)$ and respects the knot constraint requiring w and r to be of opposite parity. In this new coordinate system one may label the knot (N, w, r) by $D_{\frac{w}{2} \frac{r+o}{2}}^{N/2}(a, b, c, d)$. One thereby associates with the (N, w, r) knot a multinomial in the elements of the algebra of the form

$$D_{mm'}^j(a, b, c, d) = \sum_{\substack{\delta(n_a+n_b, n_+) \\ \delta(n_c+n_d, n_-)}} A_{mm'}^j(q, n_a, n_c) \delta(n_a + n_c, n'_+) a^{n_a} b^{n_b} c^{n_c} d^{n_d}, \tag{7.2}$$

where (j, m, m') are given by (7.1) and the explicit form of $A_{mm'}^j$ is given in (5.28). We shall also assume that q is real.

Like the Kauffman and Jones polynomials, these forms are based on the algebra of the classical knot. They are operator expressions that may be evaluated on the state space of the algebra.

Let us next compute a basis in this space.

Since b and c commute, they have common eigenstates. Let $|0\rangle$ be designated as a ground state and let

$$b|0\rangle = \beta|0\rangle, \tag{7.3}$$

$$c|0\rangle = \gamma|0\rangle, \tag{7.4}$$

$$bc|0\rangle = \beta\gamma|0\rangle. \tag{7.5}$$

From the algebra we see that

$$bc|n\rangle = E_n|n\rangle, \tag{7.6}$$

where

$$|n\rangle \sim d^n|0\rangle \tag{7.7}$$

and

$$E_n = q^{2n} \beta\gamma. \tag{7.8}$$

The eigenvalue spectrum resembles that of a harmonic oscillator but the levels are arranged in a geometrical rather than arithmetical progression. We shall refer to this spectrum as the q -oscillator spectrum.

Here d and a are raising and lowering operators respectively:

$$d|n\rangle = \lambda_n|n+1\rangle, \tag{7.9}$$

$$a|n\rangle = \mu_n|n-1\rangle. \tag{7.10}$$

Then

$$ad|n\rangle = a\lambda_n|n+1\rangle = \lambda_n\mu_{n+1}|n\rangle, \tag{7.11}$$

$$da|n\rangle = d\mu_n|n-1\rangle = \mu_n\lambda_{n-1}|n\rangle. \tag{7.12}$$

By the algebra (A), (7.11) and (7.12) become

$$(1+qbc)|n\rangle = \lambda_n\mu_{n+1}|n\rangle, \tag{7.13}$$

$$(1+q_1bc)|n\rangle = \mu_n\lambda_{n-1}|n\rangle. \tag{7.14}$$

If there is both a highest state M , and a lowest state M' , then

$$\lambda_M = \mu_{M'} = 0, \quad M' < M \tag{7.15}$$

and by (7.13) and (7.14)

$$(1+qbc)|M\rangle = 0, \tag{7.16}$$

$$(1+q_1bc)|M'\rangle = 0. \tag{7.17}$$

Then by (7.6) and (7.8)

$$q^{2M+1}\beta\gamma = q^{2M'-1}\beta\gamma \tag{7.18}$$

or

$$(q^2)^{M-M'+1} = 1. \tag{7.19}$$

Since we assume that q is real,

$$M' = M + 1. \tag{7.20}$$

Since (7.15) and (7.20) are contradictory, there may be either a highest or a lowest state but not both. The same discussion may be given for the $SU_q(2)$ algebra.

In the next section we shall postulate that the individual states of excitation of the quantum knot are represented by $D_{mm'}^j|n\rangle$. If the empirical evidence restricts the number of states, there must be an externally required physical boundary condition to cut off the otherwise infinite spectrum that is formally allowed by the $SL_q(2)$ algebra. In general the physical interpretation of $|n\rangle$ depends on the context.

8. The Quantum Knot

Since the writhe and rotation of a classical knot are topological invariants, they do not depend on the size or the shape of the knot; i.e. they are conformal invariants that hold for microscopic knots as well. These topological constraints are also to be understood here as kinematical constraints on the allowed equations of motion.

Then w and r are integrals of the motion for microscopic classical knots with spectra determined by the topology of the knot.

We shall now introduce the quantum knot by interpreting $D_{mm'}^j(a, b, c, d)$ as the kinematical description of a quantum state, where

$$\boxed{(j, m, m') = \frac{1}{2}(N, w, r + o)} \tag{8.1}$$

and (N, w, r) describes the 2d projection of the corresponding classical knot. We set $o = 1$ for the quantum trefoil. The odd integer o is a new quantum number that may assume other values for other quantum knots. We also introduce the “quantum rotation” $\tilde{r} \equiv r + o$, where o appears as a “zero-point rotation” of the quantum knot. Since the spectra of (j, m, m') are restricted by $SLq(2)$, and the spectra of (N, w, r) are restricted by knot topology, the states of the quantized knot are thus jointly restricted by both $SLq(2)$ and the knot topology. The equation (8.1) establishes a correspondence between a quantized knot described by $D_{\frac{w}{2} \frac{r+o}{2}}^{N/2}$ and the 2d-projection of a classical knot.

For the 2d-projection of the trefoil knot there are four choices of (w, r) , namely $(3, 2)$, $(-3, 2)$, $(3, -2)$, $(-3, -2)$. Regarded as 3d-classical knots, only two of these trefoils are topologically distinct; but we shall consider all four choices of $D_{\frac{w}{2} \frac{r+1}{2}}^{3/2}$ as distinct quantum states. In the following when (w, r) refers to the quantum knot, both w and r may have either sign.

One may somewhat similarly define the eigenstates of the spherical top as irreducible representations of $O(3)$ by $D_{mm'}^j(\alpha, \beta, \gamma)$ where the indices (j, m, m') refer to the angular momentum of the top and the arguments (α, β, γ) to its orientation. It is also possible to define the eigenstates of the hydrogen atom as irreducible representations of $O(3)$, expressed as $D_{mm'}^j(a_1, a_2, a_3)$ where in this case (a_1, a_2, a_3) are three coordinates on the group space of $O(3)$, and where $(2j + 1, m, m')$ are respectively the principle quantum number, the z -component of the angular momentum and the z -component of the Runge–Lenz vector.⁹ Here the quantum knot is similarly described, but it is defined on the $SLq(2)$ algebra, which is not a group.

Equation (8.1) is the “correspondence principle” of the model where (j, m, m') describes the quantum knot and (N, w, r) refers to the 2d-projection of the corresponding classical knot. The so-defined quantum knot is a 2d-image of a classical knot.

9. Toward the Lagrangian of the Knotted Standard Model

To construct the Lagrangian of the knotted standard model we first describe a mapping between the trefoil quantum knots and the elementary fermions as they are described in the standard model. We begin by replacing every left chiral field

operator, Ψ , of the standard model by the “knotted field operator,” $\Psi_{mm'}^j$, where

$$\Psi_{mm'}^j = \hat{\Psi}_{mm'}^j(x) D_{mm'}^j(a, b, c, d). \tag{9.1}$$

The right chiral field operator and the vector field operator will be introduced later.

Since $D_{mm'}^j$ lies in the $SL_q(2)$ algebra, (9.1) adds new degrees of freedom to the field quanta. Then after Ψ is replaced with $\Psi_{mm'}^j$ by (9.1) in the Lagrangian of the standard model, the space–time factor $\hat{\Psi}_{mm'}^j(x)$ will be determined by a new Lagrangian containing form factors generated by $D_{mm'}^j$. Then $\hat{\Psi}_{mm'}^j(x)$ will have an induced but clear dependence on (j, m, m') .

Under $U_a \times U_b$ transformations of the algebra (A) the new field operators transform as follows:

$$\Psi_{mm'}^j \rightarrow \hat{\Psi}_{mm'}^j(x) D_{mm'}^j(a', b', c', d') \tag{9.2}$$

or by (6.6b)

$$\Psi_{mm'}^j \rightarrow U_m \times U_{m'} \Psi_{mm'}^j. \tag{9.3}$$

For physical consistency the new field action must be invariant under (9.3), since (9.3) is induced by $U_a \times U_b$ transformations that leave the defining algebra (A) unchanged. There are then Noether charges associated with U_m and $U_{m'}$ that may be described as writhe and rotation charges, Q_w and Q_r , since $m = \frac{w}{2}$ and $m' = \frac{1}{2}(r + o)$ for quantum knots.

For quantum trefoils we set $o = 1$ and, and we now define

$$Q_w \equiv -k_w m \left(\equiv -k_w \frac{w}{2} \right), \tag{9.4}$$

$$Q_r \equiv -k_r m' \left(\equiv -k_r \frac{1}{2}(r + 1) \right), \tag{9.5}$$

where k_w and k_r are undetermined constants with dimensions of electric charge. We assume that $k_w = k_r = k$ is a universal constant with the same value for all trefoils.

The knot picture of the elementary particles is more plausible if the simplest particles are the simplest knots. We therefore consider the possibility that the most elementary fermions with electroweak isotopic spin $t = \frac{1}{2}$ are the most elementary quantum knots, the quantum trefoils with $N = 3$ and $o = 1$. This possibility is supported by the following empirical observation

$$\boxed{(t, -t_3, -t_0)_L = \frac{1}{6}(N, w, r + 1)}, \tag{9.6}$$

where t_0 is the electroweak $U(1)$ hypercharge. Equation (9.6) is satisfied by the four (left chiral) families of the elementary fermions described by $(\frac{1}{2}, t_3, t_0)_L$ and the four quantum trefoils described by $(3, w, r)$ and shown by the *row-to-row proportionality* in Table 1, as expressed in (9.6).

Table 1. Empirical support for (9.6).

(f_1, f_2, f_3)	t	t_3	t_0	$D_{\frac{w}{2}, \frac{r+1}{2}}^{N/2}$	N	w	r	$r + 1$
leptons $\left\{ \begin{array}{l} (e, \mu, \tau)_L \\ (\nu_e, \nu_\mu, \nu_\tau)_L \end{array} \right.$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$D_{\frac{3}{2}, \frac{3}{2}}^{3/2}$	3	3	2	3
	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$D_{-\frac{3}{2}, \frac{3}{2}}^{3/2}$	3	-3	2	3
quarks $\left\{ \begin{array}{l} (d, s, b)_L \\ (u, c, t)_L \end{array} \right.$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$D_{\frac{3}{2}, -\frac{1}{2}}^{3/2}$	3	3	-2	-1
	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$D_{-\frac{3}{2}, -\frac{1}{2}}^{3/2}$	3	-3	-2	-1

Only for the particular row-to-row correspondence shown in Table 1 does (9.6) hold, i.e. each of the four families of fermions labeled by (t_3, t_0) is *uniquely correlated with a specific (w, r) trefoil, and therefore with a specific $D_{mm'}^{3/2}$ quantum knot.*

Note also that with this same correspondence all the leptons (regarding the neutrinos as uncharged leptons) correspond to trefoils with positive knot helicity ($r = 2$), while the quarks correspond to trefoils of opposite knot helicity ($r = -2$).

By (8.1) and (9.6) one also has

$$(j, m, m') = 3(t, -t_3, -t_0) \tag{9.7}$$

for the left chiral fields and quantum trefoils.

In the knot model quantum knots are jointly defined by the topological condition (8.1) and the empirical constraint (9.7). Both (8.1) and (9.7) refer to the left chiral components of fermionic field operators.

In Table 2 we next compare the electric charges Q_e of the elementary fermions with the total Noether charges of the corresponding quantum trefoils. To construct and interpret this table we have again postulated that $k = k_w = k_r$ is a universal constant with the same value for all trefoils. We then obtain the value of k by

Table 2. Electric charges of leptons, quarks and quantum trefoils.

Standard model					Quantum trefoil model				
(f_1, f_2, f_3)	t	t_3	t_0	Q_e	(w, r)	$D_{\frac{w}{2}, \frac{r+1}{2}}^{N/2}$	Q_w	Q_r	$Q_w + Q_r$
$(e, \mu, \tau)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	(3, 2)	$D_{\frac{3}{2}, \frac{3}{2}}^{3/2}$	$-k(\frac{3}{2})$	$-k(\frac{3}{2})$	$-3k$
$(\nu_e, \nu_\mu, \nu_\tau)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	(-3, 2)	$D_{-\frac{3}{2}, \frac{3}{2}}^{3/2}$	$-k(-\frac{3}{2})$	$-k(\frac{3}{2})$	0
$(d, s, b)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	(3, -2)	$D_{\frac{3}{2}, -\frac{1}{2}}^{3/2}$	$-k(\frac{3}{2})$	$-k(-\frac{1}{2})$	$-k$
$(u, c, t)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	(-3, -2)	$D_{-\frac{3}{2}, -\frac{1}{2}}^{3/2}$	$-k(-\frac{3}{2})$	$-k(-\frac{1}{2})$	$2k$
$Q_e = e(t_3 + t_0)$					$Q_w = -k\frac{w}{2} \quad Q_r = -k\frac{r+1}{2}$				

requiring that the total Noether charge, $Q_w + Q_r$, of the quantum trefoil $D_{\frac{3}{2}, \frac{3}{2}}^{3/2}$ satisfies

$$Q_w + Q_r = Q_e, \tag{9.8}$$

where Q_w and Q_r are the writhe and rotation charges, and Q_e is the electric charge of the corresponding family of elementary fermions, the charged leptons as shown in Table 2.

One sees that (9.8) holds not only for charged leptons, but also for neutrinos and for both up and down quarks if

$$k = \frac{e}{3}, \tag{9.9}$$

and one also sees that t_3 and t_0 measure the writhe and rotation charges respectively:

$$Q_w = et_3 \left(= -\frac{e}{3} m = -\frac{e}{6} w \right), \tag{9.10}$$

$$Q_r = et_0 \left(= -\frac{e}{3} m' = -\frac{e}{6} (r + 1) \right). \tag{9.11}$$

Then (9.8) becomes by (9.10) and (9.11) an alternative statement of

$$Q_e = e(t_3 + t_0). \tag{9.12}$$

Also by (9.10) and (9.11)

$$Q_e = -\frac{e}{3}(m + m'), \tag{9.13}$$

or

$$Q_e = -\frac{e}{6}(w + r + 1) \tag{9.14}$$

for the quantum trefoils.^{10,11}

Then the electric charge is a measure of the writhe + rotation, of the trefoil. The total electric charge in this way resembles the total angular momentum and total magnetic moment as a sum of two parts where the localized contribution of the writhe to the charge corresponds to the localized contribution of the spin to the angular momentum and magnetic moment. In (9.14) o contributes a “zero-point charge.”

As here defined, quantum knots carry the charge expressed as both (9.12) and (9.13). The (t_3, t_0) measures of charge are based on $SU(2) \times U(1)$ while the (m, m') measures of charge are based on $SL_q(2)$. These two different measures are related at the $j = \frac{3}{2}$ level by (9.7). We shall next attempt to extend these results beyond $j = \frac{3}{2}$, and in particular to $j = \frac{1}{2}$.^{10,11}

10. The Physical Interpretation of $D_{mm'}^j$

We shall try to give physical meaning to the defining expression (7.2) for $D_{mm'}^j$ by postulating that a, b, c, d are creation operators for fermionic preons, since these are the four elements of the fundamental ($j = \frac{1}{2}$) representation given first in (4.2) and again in the following (10.1):

$$D_{mm'}^{1/2} = \begin{array}{c|cc} & m' & \\ \hline m & \frac{1}{2} & -\frac{1}{2} \\ \hline & a & b \\ \hline & c & d \end{array} \quad (10.1)$$

By (10.1) and (9.13) there is one charged preon, a , with charge $-\frac{e}{3}$ and its antiparticle, d , and there is one neutral preon, b , with its antiparticle, c .

By (8.1) the corresponding a, b, c, d classical realizations cannot be described as knots since they have only a single crossing. They can, however, be described as 2d-projections of twisted loops with $N = 1$, $w = \pm 1$ and $r = 0$. We shall propose a physical meaning for these twisted loops by interpreting them as flux tubes, and we shall regard a, b, c, d as creation operators for either preonic particles or preonic 2d-projections of flux tubes, depending on whether we assume that they concentrate energy and momentum at a point or on a curve. We may assume that the direction of flow in the flux tube defines its helicity.

Every $D_{mm'}^j$, as given in (5.27), being a polynomial in a, b, c, d , can be interpreted as a creation operator for creating a superposition of states, each state with n_a, n_b, n_c, n_d preons. The a, b, c, d population of each of these states is constrained by the triplet (j, m, m') that allows (n_a, n_b, n_c, n_d) to vary but fixes (t, t_3, t_0) and $(N, w, r + o)$ according to (9.7) and (8.1).

It then turns out that the creation operators for the charged leptons, $D_{\frac{3}{2}\frac{3}{2}}^{3/2}$, neutrinos, $D_{-\frac{3}{2}\frac{3}{2}}^{3/2}$, down quarks, $D_{\frac{3}{2}-\frac{1}{2}}^{3/2}$ and up quarks, $D_{-\frac{3}{2}-\frac{1}{2}}^{3/2}$, as required by Tables 1 and 2, are represented by (5.27) as the following monomials

$$D_{\frac{3}{2}\frac{3}{2}}^{3/2} \sim a^3, \quad D_{-\frac{3}{2}\frac{3}{2}}^{3/2} \sim c^3, \quad D_{\frac{3}{2}-\frac{1}{2}}^{3/2} \sim ab^2, \quad D_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \sim cd^2 \quad (10.2)$$

implying that charged leptons and neutrinos are composed of three a -preons and three c -preons, respectively, while the down quarks are composed of one a - and two b -preons, and the up quarks are composed of one c - and two d -preons. Both (10.1), with (9.13), and (10.2) are in agreement with the Harari-Shupe model of quarks,^{12,13} and with the experimental evidence on which their model is constructed.

To achieve the required $U_a(1) \times U_b(1)$ invariance of the knotted Lagrangian (and the associated conservation of t_3 and t_0 , or equivalently of the writhe and rotation charge), it is necessary to impose (8.1) and (9.7) on the knotted vector bosons by which the knotted fermions interact as well as on the knotted fermions themselves.¹⁵ For these electroweak vectors we assume the standard $t = 1$ and

Table 3. Electroweak vectors ($j = 3$).

	Q	t	t_3	t_0	$D_{-3t_3-3t_0}^{3t}$
W^+	e	1	1	0	$D_{-3,0}^3 \sim c^3 d^3$
W^-	$-e$	1	-1	0	$D_{3,0}^3 \sim a^3 b^3$
W^3	0	1	0	0	$D_{0,0}^3 \sim f_3(bc)$

therefore $j = 3$ and $N = 6$ since $(j, m, m') = 3(1, -t_3, -t_0)$, in accord with (9.7) and (8.1) and as shown in Table 3.

The neutral vector W_μ^3 is the superposition of four states of six preons given by

$$D_{00}^3 = A(0, 3)b^3c^3 + A(1, 2)ab^2c^2d + A(2, 1)a^2bcd^2 + A(3, 0)a^3d^3$$

according to (7.2) which is reducible by the algebra A to a function of the neutral operator bc . It is assumed that the W^0 neutral vector coupled to the hypercharge is an $SLq(2)$ singlet.

The previous considerations are based on electroweak physics. To describe the strong interactions it is necessary according to the standard model to introduce $SU(3)$. In the $SLq(2)$ electroweak model, as here described, the need for the additional $SU(3)$ symmetry appears already at the level of the charged leptons and neutrinos since they are presented in the $SLq(2)$ model at the electroweak level as a^3 and c^3 , respectively. Then the simple way to protect the Pauli principle is to make the replacements of (a, c) by (a_i, c_i) and

$$\begin{aligned} \text{charged leptons: } & a^3 \rightarrow \varepsilon^{ijk} a_i a_j a_k, \\ \text{neutrinos: } & c^3 \rightarrow \varepsilon^{ijk} c_i c_j c_k, \end{aligned}$$

where a_i and c_i provide a basis for the fundamental representation of $SU(3)$. Then the charged leptons and neutrinos are color singlets. If the b and d preons are also color singlets, then down quarks $a_i b^2$ and up quarks $c_i d^2$ provide a basis for the fundamental representation of $SU(3)$, as required by the standard model.¹⁴

We do not depart from $SLq(2)$ in the above way of introducing $SU(3)$. If one instead goes over to $SUq(2)$, where $\bar{a} = d$ and $\bar{c} = -q_1 b$, we may make use of the two complex representations 3 and $\bar{3}$ of $SU(3)$, by assigning a_i and c_i to the $\bar{3}$ and b^i and d^i to the 3 representation.²⁴

The leptons and quarks, which are associated in Table 1 with opposite knot helicity (r), are $SU(3)$ singlets and $SU(3)$ triplets, respectively, for gluon interactions. Likewise one sees from Table 4 that the $SU(3)$ triplets, a_i and c_i , have positive preon helicity ($\tilde{r} = +1$), while the $SU(3)$ singlets, b and d , have opposite preon helicity ($\tilde{r} = -1$). The association of the $SU(3)$ representation with knot and preon helicity repeats the similar association of $SU(2)$ doublets and $SU(2)$ singlets with left and right chirality, respectively, for electroweak interactions.

11. Complementarity

The representation of $D_{mm'}^j$ as a function of (a, b, c, d) and (n_a, n_b, n_c, n_d) by Eq. (5.27) implies the following constraints on the exponents:

$$n_a + n_b + n_c + n_d = 2j, \tag{11.1}$$

$$n_a + n_b - n_c - n_d = 2m, \tag{11.2}$$

$$n_a - n_b + n_c - n_d = 2m'. \tag{11.3}$$

The two relations giving physical meaning to $D_{mm'}^j$, namely (8.1) and (9.7):

$$(j, m, m') = \frac{1}{2}(N, w, r + o) \tag{11.4}$$

and

$$(j, m, m') = 3(t, -t_3, -t_0) \tag{11.5}$$

imply two different interpretations of the relations (11.1)–(11.3). By (11.4) one has

$$N = n_a + n_b + n_c + n_d, \tag{11.6}$$

$$w = n_a + n_b - n_c - n_d, \tag{11.7}$$

$$\tilde{r} \equiv r + o = n_a - n_b + n_c - n_d. \tag{11.8}$$

In (11.8), where $\tilde{r} \equiv r + o$ and o is the parity index, \tilde{r} may be termed “the quantum rotation,” and o the “zero-point rotation.”

By (11.5) one has

$$t = \frac{1}{6}(n_a + n_b + n_c + n_d), \tag{11.9}$$

$$t_3 = -\frac{1}{6}(n_a + n_b - n_c - n_d), \tag{11.10}$$

$$t_0 = -\frac{1}{6}(n_a - n_b + n_c - n_d). \tag{11.11}$$

These relations hold for all representations allowed by the model. For the elements of the fundamental representation they imply Tables 4 and 5 describing the fermionic preons.

Table 4. Elements of $j = \frac{1}{2}$ representation as twisted loops.

p	N_p	w_p	\tilde{r}_p
a	1	1	1
b	1	1	-1
c	1	-1	1
d	1	-1	-1

Table 5. Elements of $j = \frac{1}{2}$ representation as point particles.

p	t_p	t_{3p}	t_{0p}	Q_p
a	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{e}{3}$
b	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0
c	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0
d	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{e}{3}$

Equation (11.5) describes the electroweak indices (t, t_3, t_0) if $j = \frac{3}{2}$; in general, however, in this model including $j = \frac{1}{2}$ and Table 5, the (t, t_3, t_0) may be understood as code for (j, m, m') , i.e. as indices for SLq(2), not for $SU(2) \times U(1)$. Then the index t_3 in (11.5) measures writhe charge and t_0 measures rotation hypercharge as Noether charges following from the $U_a \times U_b$ invariance of the knot Lagrangian. Then t no longer needs to be integral or half-integral. The “SLq(2) knot charge” defines charge more naturally in the knot model than electroweak isotopic charge with which it agrees at $j = \frac{3}{2}$. At the $j = \frac{3}{2}$ level the $SU(2) \times U(1)$ measure requires the assumption of fractional charges for the quarks and the SLq(2) measure requires at the $j = \frac{1}{2}$ level the replacement of the fundamental charge (e) for charged leptons by a new fundamental charge ($e/3$) for charged preons. The SLq(2), or (j, m, m') measure, has a direct physical interpretation since $(j, m, m') = \frac{1}{2}(N, w, r+o)$, where $2j$ is the number of preonic sources, while $2m$ and $2m'$ respectively measure the writhe and rotation sources of charge.

In Eqs. (11.6)–(11.8) by Table 4, the numerical coefficients may be replaced by (N_p, w_p, \tilde{r}_p) as follows:

$$N = \sum_p n_p N_p, \quad p = (a, b, c, d), \tag{11.12}$$

$$w = \sum_p n_p w_p, \tag{11.13}$$

$$\tilde{r} = \sum_p n_p \tilde{r}_p, \tag{11.14}$$

and in Eqs. (11.9)–(11.11) by Table 5, the numerical coefficients may be replaced by (t_p, t_{3p}, t_{0p}) as follows:

$$t = \sum_p n_p t_p, \quad p = (a, b, c, d), \tag{11.15}$$

$$t_3 = \sum_p n_p t_{3p}, \tag{11.16}$$

$$t_0 = \sum_p n_p t_{0p}. \tag{11.17}$$

Since $r = 0$ for preonic loops, o plays the role of a quantum rotation for preons:

$$\tilde{r}_p = o_p, \quad p = (a, b, c, d). \quad (11.18)$$

For the elementary fermions presently observed,

$$\tilde{r} = r + 1. \quad (11.19)$$

We shall now regard $D_{mm'}^j$ as the creation operator for a superposition of quantum states that may be described as either the 2d-projections of knotted fields (N, w, \tilde{r}) composed of 2d-projections of preonic flux lines according to (11.12)–(11.14) or as composite particles (t, t_3, t_0) composed of preonic particles according to (11.15)–(11.17). As formal algebraic relations (11.12)–(11.17) express properties of the higher representations as additive compositions of the fundamental representation.

The representation of the four trefoils as composed of three overlapping preon loops is shown in Fig. 1. In interpreting Fig. 1, note that the two lobes of all the preons make opposite contributions to the rotation, r , so that the total rotation of each preon vanishes. When the three a -preons and c -preons are combined to form charged leptons and neutrinos, respectively, each of the three labeled circuits is counterclockwise and contributes $+1$ to the rotation while the single unlabeled and shared (overlapping) circuit is clockwise and contributes -1 to the rotation so that the total r for both charged leptons and neutrinos is $+2$. For the quarks the three labeled loops contribute -1 and the shared loop $+1$ so that $r = -2$. It is the quantum rotation (\tilde{r}), however, and not the classical rotation (r) that satisfies (11.14).

In interpreting the 2d-projections shown in Figs. 1 and 2 and other 2d-projections shown elsewhere in the model, we continue to observe the rule that 3d-motions, which would be allowed by ambient isotopy, such as topological motions that would unroll a twisted loop, are strictly forbidden.

Equation (11.6) states that the total number of preons, N' , equals the number of crossings, N . Since we assume that the preons are fermions, the knot describes a fermion or a boson depending on whether the number of crossings is odd or even. Viewed as a knot, a fermion becomes a boson when the number of crossings is changed by attaching or removing a curl. This picture is consistent with the view of a curl as an opened preon loop.

Since a and d are antiparticles with opposite charge and hypercharge, while b and c are neutral antiparticles with opposite values of the hypercharge, we may introduce the preon numbers

$$\nu_a = n_a - n_d, \quad (11.20)$$

$$\nu_b = n_b - n_c. \quad (11.21)$$

Then (11.7) and (11.8) may be rewritten as

$$\nu_a + \nu_b = w (= -6t_3), \quad (11.22)$$

$$\nu_a - \nu_b = \tilde{r} (= -6t_0). \quad (11.23)$$

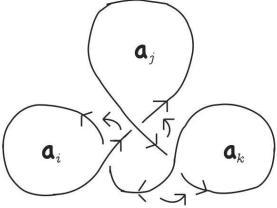

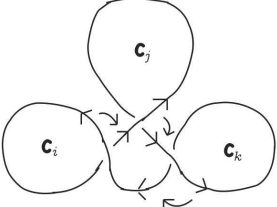

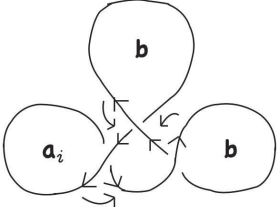

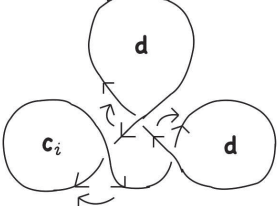

<p>Charged Leptons, $D_{\frac{3}{2}\frac{3}{2}\frac{3}{2}}^{3/2} \sim a^3$</p>  <p>$(3, 2, 1)$</p>	<p>a-preons, $D_{\frac{1}{2}\frac{1}{2}}^{1/2}$</p>  <p>$(1, 0, 1)$</p>
<p>Neutrinos, $D_{-\frac{3}{2}\frac{3}{2}\frac{3}{2}}^{3/2} \sim c^3$</p>  <p>$(-3, 2, 1)$</p>	<p>c-preons, $D_{-\frac{1}{2}\frac{1}{2}}^{1/2}$</p>  <p>$(-1, 0, 1)$</p>
<p>d-quarks, $D_{\frac{3}{2}\frac{3}{2}-\frac{1}{2}}^{3/2} \sim ab^2$</p>  <p>$(3, -2, 1)$</p>	<p>b-preons, $D_{\frac{1}{2}\frac{1}{2}}^{1/2}$</p>  <p>$(1, 0, -1)$</p>
<p>u-quarks, $D_{-\frac{3}{2}-\frac{3}{2}-\frac{1}{2}}^{3/2} \sim cd^2$</p>  <p>$(-3, -2, 1)$</p>	<p>d-preons, $D_{-\frac{1}{2}-\frac{1}{2}}^{1/2}$</p>  <p>$(-1, 0, -1)$</p>

Fig. 1. Preonic structure of elementary fermions $Q = -\frac{e}{6}(w + r + o)$, and $(j, m, m') = \frac{1}{2}(N, w, r + o)$.

By (11.22) and (11.23) the conservation of the preon numbers and of the charge and hypercharge is equivalent to the conservation of the writhe and rotation, which are topologically conserved at the 2d-classical level. In this respect, these quantum conservation laws correspond to the classical conservation laws.

One may view the symmetry of an elementary particle, defined by representations of the $SLq(2)$ algebra, in any of the following ways:

$$D_{mm'}^j = D_{-3t_3-3t_0}^{3t} = D_{\frac{w}{2}, \frac{\tilde{r}}{2}}^{N/2} = \tilde{D}_{\nu_a \nu_b}^{N'} \quad (11.24)$$

where N' is the total number of preons.

The point particle (N', ν_a, ν_b) representation and the flux loop (N, w, \tilde{r}) complementary representation are related by

$$\tilde{D}_{\nu_a \nu_b}^{N'} = \sum_{N, w, r} \delta(N', N) \delta(\nu_a + \nu_b, w) \delta(\nu_a - \nu_b, \tilde{r}) D_{\frac{w}{2}, \frac{\tilde{r}}{2}}^{N/2}. \quad (11.25)$$

In (11.25) the correspondence between classical and quantum knots introduced in (8.1) has here been promoted to a correspondence between fields and particles.

Since one may interpret the elements (a, b, c, d) of the $SLq(2)$ algebra as creation operators for either preonic particles or flux loops, the D_{mp}^j may be interpreted as a creation operator for a composite particle composed of either preonic particles or flux loops. *These two complementary views of the same particle may be reconciled as describing N -body systems bound by a knotted field having N -crossings as illustrated in Fig. 2 for $N = 3$.* In the limit where the three outside lobes become infinitesimal compared to the central circuit, the resultant structure will resemble a three particle system tied together by a string. The $j = 1$ representation does not play an explicit role in the picture just described. In a different physical interpretation of the algebra, the $j = 1$ vector field binds the three $j = \frac{1}{2}$ preons to form the $j = \frac{3}{2}$ elementary fermions.

On the other hand, in an alternative interpretation of complementarity, the hypothetical preons conjectured to be present in Fig. 2 carry no independent degrees of freedom and may simply describe concentrations of energy, momentum and charge at the crossings of the flux tube. In this interpretation of complementarity, (t, t_3, t_0) and (N, w, \tilde{r}) are just two ways of describing the same quantum trefoil of field. In this picture the preons are bound, i.e. they do not appear as free particles.

If $j = 0$, the indices of the quantum knot are

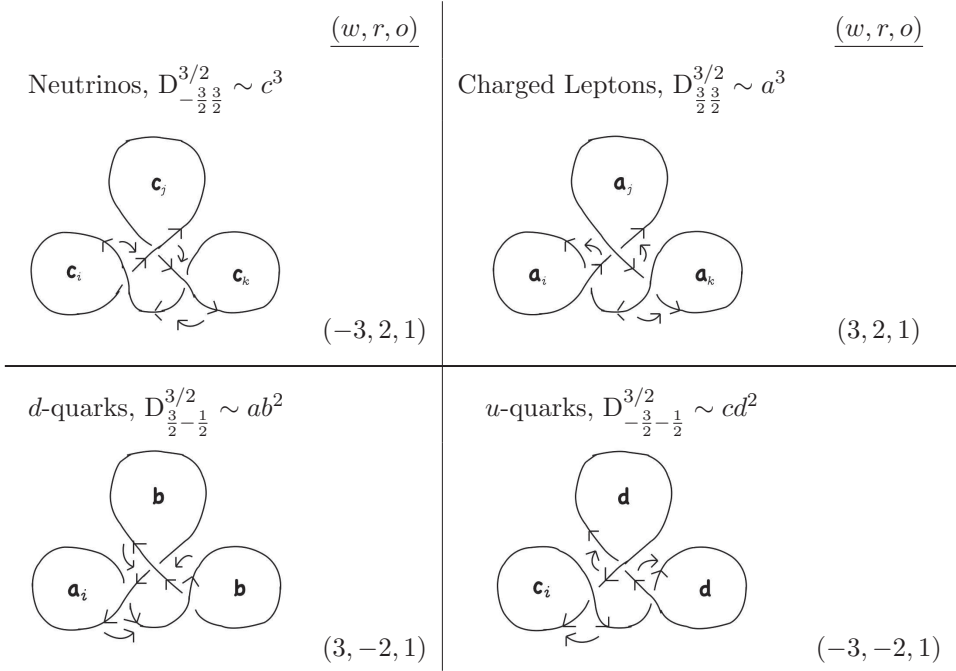
$$(j, m, m') = (0, 0, 0) \quad (11.26)$$

and by the rules in (11.4) and (11.5) for interpreting the knot indices on the left chiral fields

$$(N, w, \tilde{r}) = (0, 0, 0) \quad \text{by (11.4)}, \quad (11.27)$$

$$(t, -t_3, -t_0) = (0, 0, 0) \quad \text{by (11.5)}. \quad (11.28)$$

Then by (11.28) the $j = 0$ states have no electroweak interactions and by (11.27) they are simple flux loops with no crossings ($N = 0$). It is possible that these



The preons conjectured to be present at the crossings are not shown in these figures.

Fig. 2. Leptons and quarks pictured as three preons bound by a trefoil field.

hypothetical states are realized as electroweak noninteracting unknotted loops of field flux with $r = \pm 1$ and $\tilde{r} = 0$.

If, as we are assuming, the leptons and quarks with $j = \frac{3}{2}$ correspond to 2d representations of knots with three crossings, and if the heavier preons with $j = \frac{1}{2}$ correspond to 2d representations of twisted loops with one crossing, then if the $j = 0$ states correspond to 2d projections of simple loops, one might conjecture that these particles with no electroweak interactions are smaller and heavier than the preons, and are among the candidates for “dark matter.”

Since the topological diagram of Fig. 2 describes loops that have no size or shape or associated equations of motion, we shall next introduce an explicit Lagrangian to obtain a dynamical interpretation of these figures.

12. The Knot and Preon Lagrangians¹⁵⁻¹⁷

To construct the knot Lagrangian we replace the left chiral field operators for the elementary fermions, $\Psi_L(x)$, and the electroweak vectors, $W_\mu(x)$, of the standard model by $\hat{\Psi}_L^{3/2}(x)D_{mm'}^{3/2}$, and $\hat{W}_\mu^3(x)D_{mm'}^3$, respectively. To construct the preon Lagrangian we replace $\Psi_L(x)$ and $W_\mu(x)$ by $\hat{\Psi}_L^{1/2}(x)D_{mm'}^{1/2}$, and $\hat{W}_\mu^1(x)D_{mm'}^1$, respectively. In both the knot and the preon Lagrangians we preserve the local

$SU(3) \times SU(2) \times U(1)$ symmetry and to this degree the dynamics of the standard model, but the knot factors will introduce form factors in all terms and thereby distinguish the preon from the knot Lagrangian. The factors $\hat{\Psi}_L^j(x)$ and $\hat{W}_\mu^j(x)$ in the knot and preon models record the masses and momenta of either the hypothetical composite fermions and bosons of the knot model or of the hypothetical preons and preonic vectors of the preon model. We will assume that every right $SU(2)$ singlet has the same $D_{mm'}^j$ factor as the corresponding left $SU(2)$ doublet for both $j = \frac{3}{2}$ and $j = \frac{1}{2}$; it will be clear that this assumption is required if the Higgs is knotted in the simplest way.

We take as a starting point the Lagrangian of the standard model at the electroweak level written as follows¹⁶

$$\begin{aligned} \mathcal{L}_{st} = & -\frac{1}{4} \text{Tr} W^{\mu\lambda} W_{\mu\lambda} - \frac{1}{4} H^{\mu\lambda} H_{\mu\lambda} + i[\bar{L}\nabla L + \bar{R}\nabla R] \\ & + \frac{1}{2} \overline{\nabla\varphi} \nabla\varphi - V(\bar{\varphi}\varphi) - \frac{m}{\rho} [\bar{L}\varphi R + \bar{R}\bar{\varphi}L]. \end{aligned} \quad (12.1)$$

Here $W_{\mu\lambda}$ is the non-Abelian gauge field and $H_{\mu\lambda}$ is the hypercharge part of the Weinberg–Salam gauge field; L and R are the left and right chiral components of a fermion field and φ is the Higgs field. The Yukawa coupling matrix in the mass term has here been replaced by a multiple of the unit matrix.

We replace every field operator Ψ of the standard model by

$$\Psi \rightarrow \hat{\Psi} D_{mm'}^j, \quad (12.2)$$

where the left-chiral creation operators $D_{mm'}^j$ are determined empirically, as discussed in Secs. 8 and 9, subject to the requirement that the modified action be $SU(2) \times U(1)$ and $U_a(1) \times U_b(1)$ invariant. In this Lagrangian, there are no $SU(3)$ couplings, i.e. in the following exposition the preon operators do not carry the indices needed to formulate the standard model beyond the electroweak level.

To implement (12.2) we begin with

$$\Psi(t, t_3, t_0) \rightarrow \hat{\Psi}_{mm'}^j(x | t, t_3, t_0) D_{mm'}^j, \quad (12.3)$$

where the first factor preserves the $SU(2) \times U(1)$ symmetry of the standard model and the (t, t_3, t_0) of this factor are the electroweak isotopic spin quantum numbers of the standard model. The second factor $D_{mm'}^j$ expresses the postulated knot symmetry and the (j, m, m') are the knot quantum numbers. We assume that the Higgs is an isotopic doublet as in the standard model, *and as a fundamental defining assumption in the $SLq(2)$ extension of the standard model we postulate that the knotted Higgs is an $SLq(2)$ singlet.* Then the requirement of $U_a(1) \times U_b(1)$ invariance of the mass term in (12.1) requires that every *right chiral field R has the same knot factor $D_{mm'}^j$ as the corresponding L field.* In the same way the $U_a(1) \times U_b(1)$ invariance of the fermion–boson interaction determines the $D_{mm'}^j$, carried by the boson that mediates this interaction.

For the left chiral field we examine the possible meaning of the extension of the empirically-based relation (9.7) from $j = \frac{3}{2}$ to other j if (9.7) is retained unchanged as follows

$$(j, m, m') = 3(t, -t_3, -t_0), \tag{12.4}$$

while we also retain the $SU(2) \times U(1)$ expression for charge

$$Q = e(t_3 + t_0) \tag{12.5}$$

as well as the $SLq(2)$ expression for charge

$$Q = -\frac{e}{3}(m + m') \tag{12.6}$$

but we do not assume (12.4) for the right chiral field, which is assumed to be an isotopic singlet with $t = 0$ here as in the standard model. We shall also assume that the space-time dependent factor $\hat{\Psi}_{mm'}^j(x)$ for both L and R , at both the $j = \frac{3}{2}$ (knot) level and at the $j = \frac{1}{2}$ (preon) level, retains the symmetry of the standard model except for the global form factors generated by and contributed to the Lagrangian by the $D_{mm'}^j$.

If the changes in the standard model are made according to the preceding substitutions, the new Lagrangian will be $U_a(1) \times U_b(1)$ invariant as required, so that all modified terms appearing in the new Lagrangian will be functions of bc . The new operator Lagrangian is then numerically valued on eigenstates of bc and is therefore a function of $\beta\gamma$, the eigenvalues of bc . The new field operators will appear in the new Lagrangians as dependent on (j, m, m') form factors parametrized by bc .

In the standard model, L and φ are isotopic doublets while $\bar{L}\varphi$ and R are isotopic singlets. We retain this isotopic structure in both the knot ($j = \frac{3}{2}$) and preon ($j = \frac{1}{2}$) modified versions of (12.1) and continue to follow the standard model by going to the unitary gauge where φ has a single component which is neutral. Since we have assumed that φ is an $SLq(2)$ singlet and that L and R carry the same $D_{mm'}^j$, the mass term in (12.1) is $U_a(1) \times U_b(1)$ invariant as required.

In the interaction terms as well as in the mass terms we retain the organization of the four fermionic fields of the standard model into two $SU(2)$ doublets.

The knotted isotopic or writhe doublets at the level of the standard model are

$$L(\nu, \ell) = \begin{pmatrix} \hat{\Psi}_L(\nu)D_{-\frac{3}{2}\frac{3}{2}}^{3/2} \\ \hat{\Psi}_L(\ell)D_{\frac{3}{2}\frac{3}{2}}^{3/2} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\Psi}_L(\nu)c^3 \\ \hat{\Psi}_L(\ell)a^3 \end{pmatrix}, \quad \begin{array}{c|cc} & t_3 & Q \\ \hline \nu & \frac{1}{2} & 0 \\ \ell & -\frac{1}{2} & -1 \end{array} \tag{12.7}$$

where l labels the charged leptons, and

$$L(u, d) = \begin{pmatrix} \hat{\Psi}_L(u)D_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \\ \hat{\Psi}_L(d)D_{\frac{3}{2}-\frac{1}{2}}^{3/2} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\Psi}_L(u)cd^2 \\ \hat{\Psi}_L(d)ab^2 \end{pmatrix}, \quad \begin{array}{c|cc} & t_3 & Q \\ \hline u & \frac{1}{2} & \frac{2}{3} \\ d & -\frac{1}{2} & -\frac{1}{3} \end{array} \tag{12.8}$$

The corresponding writhe doublets at the preon level are by Table 5

$$L(c, a) = \left(\begin{array}{c} \hat{\Psi}_L(c)D^{1/2}_{-\frac{1}{2}\frac{1}{2}} \\ \hat{\Psi}_L(a)D^{1/2}_{\frac{1}{2}\frac{1}{2}} \end{array} \right) \rightarrow \left(\begin{array}{c} \hat{\Psi}_L(c)c \\ \hat{\Psi}_L(a)a \end{array} \right), \quad \begin{array}{c} t_3 \quad Q \\ \hline c \quad \frac{1}{6} \quad 0 \\ a \quad -\frac{1}{6} \quad -\frac{1}{3} \end{array} \quad (12.9)$$

and

$$L(d, b) = \left(\begin{array}{c} \hat{\Psi}_L(d)D^{1/2}_{-\frac{1}{2}-\frac{1}{2}} \\ \hat{\Psi}_L(b)D^{1/2}_{\frac{1}{2}-\frac{1}{2}} \end{array} \right) \rightarrow \left(\begin{array}{c} \hat{\Psi}_L(d)d \\ \hat{\Psi}_L(b)b \end{array} \right), \quad \begin{array}{c} t_3 \quad Q \\ \hline d \quad \frac{1}{6} \quad \frac{1}{3} \\ b \quad -\frac{1}{6} \quad 0 \end{array} \quad (12.10)$$

after the constant factors in the monomials $D^j_{mm'}$, have been dropped. Here the hypothetical x -dependent factors $(\hat{\Psi}_L(c), \hat{\Psi}_L(a))$ and $(\hat{\Psi}_L(d), \hat{\Psi}_L(b))$ describing preons are also assumed to be t_3 -doublets in the $j = \frac{1}{2}$ representation of $SU(2)$, like the field operators $(\hat{\Psi}_L(\nu), \hat{\Psi}_L(\ell))$ and $(\hat{\Psi}_L(u), \hat{\Psi}_L(d))$ of the standard model, i.e. the first factor $\hat{\Psi}_L$ preserves the $SU(2) \times U(1)$ symmetry of the standard model, while the second factor $D^j_{mm'}$, lying in the $SLq(2)$ algebra, expresses the postulated knot symmetry.

In both the knot and preon actions the Lagrangian expresses the local non-Abelian dynamics of the standard model and differ from the standard model and each other only by invariant form factors like $\bar{D}^j_{mm'}D^j_{mm'}$. These factors are $U_a \times U_b$ invariant and are functions of $SLq(2)$ parameters. The differences between the preon and knot Lagrangians then stem from the form factors generated by the $j = 1$ vectors and $j = \frac{1}{2}$ fermions in the preon Lagrangian since these differ from the form factors associated with the $j = 3$ vectors and the $j = \frac{3}{2}$ fermions in the knot Lagrangian.

13. The Mass Terms

In the mass term of (12.1) the complex 3×3 Yukawa coupling matrix of the standard model has been replaced by a multiple of the unit matrix. The masses and mixings of the quarks therefore do not arise here from the same Yukawa interactions with the Higgs condensate as they do in the standard model. Rather, in the present $SLq(2)$ modification of the standard model, the masses and mixings are determined by the knot form factors as modulated by scalar Yukawa couplings.

Let us now begin to convert (12.1) to a knot or preon Lagrangian by first expressing the Higgs mass term as a sum of four parts, either the contributions of the four elementary fermions, or the contributions of the four preons, as follows.

(a) The Knotted Mass Operator at the Level of the Standard Lagrangian

$$m_{\text{knot}} = m_l + m_\nu + m_{\text{up}} + m_{\text{down}} + \text{adjoint} \quad (13.1)$$

where

$$m_l = \bar{L}(\nu, l)\varphi(l)R(l), \tag{13.2}$$

$$m_\nu = \bar{L}(\nu, l)\varphi(\nu)R(\nu), \tag{13.3}$$

$$m_{\text{up}} = \bar{L}(u, d)\varphi(u)R(u), \tag{13.4}$$

$$m_{\text{down}} = \bar{L}(u, d)\varphi(d)R(d). \tag{13.5}$$

Here $L(\nu, l)$ and $L(u, d)$ are given by (12.7) and (12.8).

(b) The Knotted Mass Operator at the Level of the Preon Lagrangian

$$m_{\text{preon}} = m_a + m_b + m_c + m_d + \text{adjoint}, \tag{13.6}$$

where

$$m_a = \bar{L}(c, a)\varphi(a)R(a), \tag{13.7}$$

$$m_c = \bar{L}(c, a)\varphi(c)R(c), \tag{13.8}$$

$$m_b = \bar{L}(d, b)\varphi(b)R(b), \tag{13.9}$$

$$m_d = \bar{L}(d, b)\varphi(d)R(d). \tag{13.10}$$

Here $L(c, a)$ and $L(d, b)$ are given by (12.9) and (12.10). In all of these mass operators, φ is a Higgs field. Since we are proposing that the Higgs field is a knot singlet, L and R carry the same knot factor.

There are four knots of elementary fermions (l, ν, u, d) at the level of the standard electroweak theory as well as four preons at the level of the $SL_q(2)$ preon model (a, b, c, d), with both arranged in two doublets: (ν, l) and (up, down) at the knot level with (c, a) and (d, b) at the preon level.

Because of the wide spread in mass at the $j = \frac{3}{2}$ level and possibly at the $j = \frac{1}{2}$ level, we assume that there may be four Higgs factors at both the $j = \frac{3}{2}$ and the $j = \frac{1}{2}$ levels. In the unitary gauge the four Higgs doublets at the two levels are as follows:

$$\varphi_\nu = \begin{pmatrix} \rho_\nu \\ 0 \end{pmatrix}, \quad \varphi_l = \begin{pmatrix} 0 \\ \rho_l \end{pmatrix}, \quad \varphi_{\text{up}} = \begin{pmatrix} 0 \\ \rho_{\text{up}} \end{pmatrix}, \quad \varphi_{\text{down}} = \begin{pmatrix} \rho_{\text{down}} \\ 0 \end{pmatrix} \tag{13.11}$$

and

$$\varphi_a = \begin{pmatrix} 0 \\ \rho_a \end{pmatrix}, \quad \varphi_c = \begin{pmatrix} \rho_c \\ 0 \end{pmatrix}, \quad \varphi_b = \begin{pmatrix} 0 \\ \rho_b \end{pmatrix}, \quad \varphi_d = \begin{pmatrix} \rho_d \\ 0 \end{pmatrix}. \tag{13.12}$$

One then computes, for example, m_l , the mass term of the charged leptons by (13.2) and (12.7) as

$$\bar{L}(\nu, l)\varphi(l)R(l) = (\bar{\nu}_L \bar{c}^3 \bar{l}_L \bar{a}^3) \begin{pmatrix} 0 \\ \rho_l \end{pmatrix} (l_R a^3) = \rho_l \bar{a}^3 a^3 \bar{l}_L l_R \tag{13.13}$$

and the adjoint

$$\bar{R}(l)\bar{\varphi}(l)L(\nu, a) = \rho_l \bar{a}^3 a^3 \bar{l}_R l_L. \tag{13.14}$$

Here ν_L and l_L have been written for $\hat{\Psi}_L(\nu)$ and $\hat{\Psi}_L(l)$ respectively when substituting (12.7) in (13.2).

Evaluating these mass operators on a state $|n\rangle$ of the $SLq(2)$ algebra gives

$$\begin{aligned} &\langle n | \bar{L}(\nu, l) \varphi(l) R(l) + \bar{R}(l) \bar{\varphi}(l) L(\nu, l) | n \rangle \\ &= \rho_l \langle n | \bar{a}^3 a^3 | n \rangle (\bar{l}_L l_R + \bar{l}_R l_L) = \rho_l \langle n | \bar{a}^3 a^3 | n \rangle \bar{l}l \end{aligned} \quad (13.15)$$

and the charged lepton mass as a function of n is

$$m_l(n) = \rho_l \langle n | \bar{a}^3 a^3 | n \rangle. \quad (13.16)$$

One finds similar results for all the Higgs induced masses:

$$\boxed{\begin{aligned} m_l(n) &= \rho_l \langle n | \bar{a}^3 a^3 | n \rangle, & m_{\text{up}}(n) &= \rho_{\text{up}} \langle n | \bar{d}^2 \bar{c} \cdot cd^2 | n \rangle, \\ m_\nu(n) &= \rho_\nu \langle n | \bar{c}^3 c^3 | n \rangle, & m_{\text{down}}(n) &= \rho_{\text{down}} \langle n | \bar{b}^2 \bar{a} \cdot ab^2 | n \rangle \end{aligned}} \quad (13.17)$$

and

$$\boxed{\begin{aligned} m_a(n) &= \rho_a \langle n | \bar{a}a | n \rangle, & m_b(n) &= \rho_b \langle n | \bar{b}b | n \rangle, \\ m_c(n) &= \rho_c \langle n | \bar{c}c | n \rangle, & m_d(n) &= \rho_d \langle n | \bar{d}d | n \rangle. \end{aligned}} \quad (13.18)$$

It may be shown by the $SLq(2)$ algebra that the mass operators are simple polynomials in bc and may be evaluated on the states, $|n\rangle$, of the $SLq(2)$ algebra, which are eigenstates of bc . These states may be used to label the three generations or flavors of the leptons and quarks. For the preons there are no corresponding experimental suggestions and one may set $|n\rangle = |0\rangle$. *In all cases, the “bare Higgs mass,” ρ , is rescaled by $\langle n | \bar{D}_{mm'}^j D_{mm'}^j | n \rangle$.* For the “physical picture” the states $|n\rangle$ may be interpreted as representing either particles or flux loops.

Note that if the mass of the $j = 0$ particle is computed in the same way as the masses of the $j = \frac{1}{2}$ and $j = \frac{3}{2}$ particles, then the rescaling factor is absent for $j = 0$, since $D_{00}^0 = 1$, so that the mass of the $j = 0$ particle, computed in this way, is determined *entirely by the Higgs factor. It is then possible to identify the $j = 0$ particle with a Higgs particle. The “Higgs factors” may then be regarded as expectation values of four Higgs fields or as the expectation value of a single Higgs field multiplied by four Yukawa coupling constants.*

Although it is not necessary in this model to know the Higgs factors in order to compute relative masses of the three generations,^{19,20} it is still necessary to know these factors in order to compute absolute masses. It is also clear that a spectrum of values of the Higgs factor is required to agree with the observed mass spectrum of the leptons and quarks. It may then be natural in this model to conjecture a spectrum of masses for the $j = 0$ Higgs field to fix the absolute masses of the $j = \frac{3}{2}$, $\frac{1}{2}$ and 0 particles.

14. The Fermion–Boson Interaction

The fermion–boson interaction term in (12.1) is

$$i[\bar{L}\nabla L + \bar{R}\nabla R], \tag{14.1}$$

where L and R are again the left and right chiral fields, and L is the isotopic doublet described in (12.7) and (12.8) at the composite fermionic level, or by (12.9) and (12.10) at the preonic level. R is an isotopic singlet with the same knot factor as L . Here ∇ is the covariant derivative

$$\nabla = \not{\partial} + \mathcal{W}, \tag{14.2}$$

where \mathcal{W} is the vector connection.

If \mathcal{W} is the vector connection of the standard model, then

$$\mathcal{W} = ig(W^+t_+ + W^-t_- + W^3t_3) + ig_0W^0t_0. \tag{14.3}$$

To go over to the knot and preon models we follow (12.2) and therefore replace (W^+, W^-, W^3) in (14.3) by $(W^+D_{-30}^3, W^-D_{30}^3, W^3D_{00}^3)$ in the $j = 3$ representation of $SLq(2)$ and by $(w^+D_{-10}^1, w^-D_{10}^1, w^0D_{00}^1)$ in the $j = 1$ representation of $SLq(2)$. The particular realization of (j, m, m') as $3(t, -t_3, -t_0)$ in (12.2) for the vector bosons as well as for the left-chiral fermions has been chosen to satisfy the required conservation of $U_a \times U_b$ in the interaction terms.¹⁵ We may replace \mathbf{t} in (14.3) by $\boldsymbol{\tau}$ as follows:

$$\tau_{\pm}^j = c_{\pm}^j t_{\pm} \mathcal{D}_{\pm 0}^j, \quad j = 1, 3, \tag{14.4}$$

$$\tau_3^j = c_3^j t_3 \mathcal{D}_3^j, \quad j = 1, 3. \tag{14.5}$$

Here $\mathcal{D}_{\pm 0}^j$ is the operator part of the monomial $D_{\pm j 0}^j$:

$$\mathcal{D}_{\pm 0}^j = D_{\pm j 0}^j / A_{\pm j 0}^j, \quad j = 1, 3 \tag{14.6}$$

and

$$\mathcal{D}_3^j = D_{00}^j, \quad j = 1, 3 \tag{14.7}$$

D_{00}^j is a function of the operator bc , but it is not a monomial.

The determination of the constants c_k^j in (14.4) and (14.5) is deferred to the section on the kinetic energy of the Higgs.

Since we have assumed in both the $j = 3$ and $j = 1$ representations the relations (9.7) and (9.12), we have Tables 6 and 7.

In the adjoint representation where $j = 1$ and $t = +\frac{1}{3}$, the possible values of m and m' are $(1, 0, -1)$ and the corresponding values for t_3 and t_0 are $(\frac{1}{3}, 0, -\frac{1}{3})$ by (9.7) extended to $j = 1$. We assign $t_0 = 0$ to all the vector bosons. Then we have by (12.4) Tables 6 and 7 for the $j = 3$ (knotted electroweak vectors) and $j = 1$ (preonic vectors).

The charges in Table 6 are the writhe and rotation charges of $SLq(2)$. In Table 7 they are the charges of both $SU(2) \times U(1)$ and $SLq(2)$.

The $U(1)$ hypercharge field, W^0 , appearing in (14.3) will be assumed to be an $SLq(2)$ singlet.

Table 6. Adjoint (preonic) vector ($j = 1$).

	t	t_3	t_0	Q	$D_{-3t_3-3t_0}^{3t}$
w^+	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{e}{3}$	$D_{-10}^1 = cd = \mathcal{D}_+(\frac{1}{3})$
w^-	$\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{e}{3}$	$D_{10}^1 = ab = \mathcal{D}_-(\frac{1}{3})$
w^3	$\frac{1}{3}$	0	0	0	$D_{00}^1 = ad + bc = \mathcal{D}_3(\frac{1}{3})$

Table 7. Knotted (electroweak) vector ($j = 3$).

	t	t_3	t_0	Q	$D_{-3t_3-3t_0}^{3t}$
W^+	1	1	0	e	$D_{-30}^3 \sim c^3 d^3 = \mathcal{D}_+(1)$
W^-	1	-1	0	$-e$	$D_{30}^3 \sim a^3 b^3 = \mathcal{D}_-(1)$
W^3	1	0	0	0	$D_{00}^3 = f_3(bc) = \mathcal{D}_3(1)$

15. Preon Couplings in the Preon Lagrangian

In the preon Lagrangian, the left chiral interaction terms are by (12.1), (12.9) and (12.10),

$$\bar{L}\nabla L = (\bar{L}\nabla L)(c, a) + (\bar{L}\nabla L)(d, b), \quad (15.1)$$

where

$$(\bar{L}\nabla L)(c, a) = (\bar{\Psi}_L(c)\bar{c}, \quad \bar{\Psi}_L(a)\bar{a})(\not{\partial} + \mathcal{W}) \begin{pmatrix} \Psi_L(c)c \\ \Psi_L(a)a \end{pmatrix} \quad (15.2)$$

and

$$(\bar{L}\nabla L)(d, b) = (\bar{\Psi}_L(d)\bar{d}, \quad \bar{\Psi}_L(b)\bar{b})(\not{\partial} + \mathcal{W}) \begin{pmatrix} \Psi_L(d)d \\ \Psi_L(b)b \end{pmatrix}. \quad (15.3)$$

The derivative term in (15.2) may be rewritten as follows:

$$(\bar{\Psi}_L(c)\bar{c}, \quad \bar{\Psi}_L(a)\bar{a})\not{\partial} \begin{pmatrix} \Psi_L(c)c \\ \Psi_L(a)a \end{pmatrix} = \bar{c}c\bar{\Psi}_L(c)\not{\partial}\Psi_L(c) + \bar{a}a\bar{\Psi}_L(a)\not{\partial}\Psi_L(a) \quad (15.4)$$

$$= \bar{\Psi}_L(c)\not{\Delta}_c\Psi_L(c) + \bar{\Psi}_L(a)\not{\Delta}_a\Psi_L(a), \quad (15.5)$$

where $\not{\Delta}_c$ and $\not{\Delta}_a$ may be interpreted as momentum operators for c and a preons rescaled by the same factors that rescale their mass operators in (13.18):

$$\not{\Delta}_c = \bar{c}c\not{\partial} \quad (15.6c)$$

and

$$\not{\Delta}_a = \bar{a}a\not{\partial}. \quad (15.6a)$$

By (15.2) and (14.3)–(14.5) there is the following contribution of the non-Abelian gauge field

$$\begin{aligned} & (\bar{\Psi}_L(c)\bar{c}, \quad \bar{\Psi}_L(a)\bar{a})\mathcal{W} \begin{pmatrix} \Psi_L(c)c \\ \Psi_L(a)a \end{pmatrix} \\ &= (\bar{\Psi}_L(c)\bar{c}, \quad \bar{\Psi}_L(a)\bar{a}) \begin{pmatrix} c_3\mathcal{D}_3w^3 & c_+\mathcal{D}_+w^+ \\ c_-\mathcal{D}_-w^- & -c_3\mathcal{D}_3w^3 \end{pmatrix} \begin{pmatrix} \Psi_L(c)c \\ \Psi_L(a)a \end{pmatrix} \end{aligned} \quad (15.7)$$

$$\begin{aligned} &= F_{\bar{c}c}[\bar{\Psi}_L(c)w^3\Psi_L(c)] + F_{\bar{c}a}[\bar{\Psi}_L(c)w^+\Psi_L(a)] \\ &\quad + F_{\bar{a}c}[\bar{\Psi}_L(a)w^-\Psi_L(c)] - F_{\bar{a}a}[\bar{\Psi}_L(a)w^3\Psi_L(a)]. \end{aligned} \quad (15.8)$$

Here w^+ , w^- and w^3 are components of the adjoint vector field ($j = 1$) that mediate the interaction between the left chiral preons. The adjoint vector–preon form factors are

$$F_{\bar{c}c} = c_3\bar{c}\mathcal{D}_3c = c_3\bar{c}f_3(bc)c, \quad F_{\bar{c}a} = c_+\bar{c}\mathcal{D}_+a = c_+\bar{c}(cd)a, \quad (15.9)$$

$$F_{\bar{a}c} = c_-\bar{a}\mathcal{D}_-c = c_-\bar{a}(ab)c, \quad F_{\bar{a}a} = c_3\bar{a}\mathcal{D}_3a = c_3\bar{a}f_3(bc)a \quad (15.10)$$

by (14.7) and Table 6 where

$$f_3(bc) = ad + bc. \quad (15.11)$$

The form factors (15.9) and (15.10) are all invariant under $U_a \times U_b$ and are therefore expressible as functions of the knot parameters.

One may discuss (15.3) in the same way as (15.2). Beginning with the derivative term one has

$$\begin{aligned} & (\bar{\Psi}_L(d)\bar{d}, \quad \bar{\Psi}_L(b)\bar{b})\not{\partial} \begin{pmatrix} \Psi_L(d)d \\ \Psi_L(b)b \end{pmatrix} = \bar{d}d\bar{\Psi}_L(d)\not{\partial}\Psi_L(d) + \bar{b}b\bar{\Psi}_L(b)\not{\partial}\Psi_L(b) \\ &= \bar{\Psi}_L(d)\not{\Delta}_d\Psi_L(d) + \bar{\Psi}_L(b)\not{\Delta}_b\Psi_L(b), \end{aligned} \quad (15.12)$$

where $\not{\Delta}_d$ and $\not{\Delta}_b$ are momentum operators, again rescaled by the same factors that rescale the d and b preon masses in (13.18):

$$\not{\Delta}_d = \bar{d}d\not{\partial} \quad (15.13)$$

and

$$\not{\Delta}_b = \bar{b}b\not{\partial}. \quad (15.14)$$

In (15.3) one also has again the following contribution of the non-Abelian gauge field:

$$\begin{aligned} & (\bar{\Psi}_L(d)\bar{d}, \quad \bar{\Psi}_L(b)\bar{b})\mathcal{W} \begin{pmatrix} \Psi_L(d)d \\ \Psi_L(b)b \end{pmatrix} \\ &= (\bar{\Psi}_L(d)\bar{d}, \quad \bar{\Psi}_L(b)\bar{b}) \begin{pmatrix} c_3\mathcal{D}_3w^3 & c_+\mathcal{D}_+w^+ \\ c_-\mathcal{D}_-w^- & -c_3\mathcal{D}_3w^3 \end{pmatrix} \begin{pmatrix} \Psi_L(d)d \\ \Psi_L(b)b \end{pmatrix} \end{aligned} \quad (15.15)$$

$$\begin{aligned}
 &= F_{\bar{a}d} [\bar{\Psi}_L(d) w^3 \Psi_L(d)] + F_{\bar{a}b} [\bar{\Psi}_L(d) w^+ \Psi_L(b)] \\
 &\quad + F_{\bar{b}d} [\bar{\Psi}_L(b) w^- \Psi_L(d)] - F_{\bar{b}b} [\bar{\Psi}_L(b) w^3 \Psi_L(b)], \tag{15.16}
 \end{aligned}$$

where

$$\begin{aligned}
 F_{\bar{a}d} &= c_3 \bar{d} \mathcal{D}_3 d, & F_{\bar{a}b} &= c_+ \bar{d} \mathcal{D}_+ b, \\
 F_{\bar{b}d} &= c_- \bar{b} \mathcal{D}_- d, & F_{\bar{b}b} &= c_3 \bar{b} \mathcal{D}_3 b.
 \end{aligned} \tag{15.17}$$

The operator form factors

$$\bar{d} \mathcal{D}_+ b = \bar{d}(cd)b \tag{15.18}$$

and

$$\bar{b} \mathcal{D}_- d = \bar{b}(ab)d \tag{15.19}$$

are invariant under $U_a \times U_b$ as required, and by (15.11) the elements $F_{\bar{a}d}$ and $F_{\bar{b}b}$ are also invariant. All the invariant form factors are simple functions of bc .

16. Knot Couplings in the Knot Lagrangian

The non-Abelian part of the fermion–boson interaction in the knot Lagrangian is

$$\sum_i \bar{L}(i) \nabla L(i), \tag{16.1}$$

where L , R and ∇ are now all lying in the $SLq(2)$ algebra, and the sum over i is over the two doublets described by (12.7) and (12.8). The only modification of ∇ in going over to the knot model is the replacement of \mathfrak{t} by τ .

We next consider the detailed dependence of (16.1) on knot form factors. For the charged lepton–neutrino doublet we have, dropping the Feynman slash, and denoting the knotted L and \mathcal{W} by \hat{L} and $\hat{\mathcal{W}}$,

$$\bar{L} \nabla L \Rightarrow (\bar{\nu} \quad \bar{l})_{\hat{L}} (\partial + ig \hat{\mathcal{W}}) \begin{pmatrix} \nu \\ l \end{pmatrix}_{\hat{L}}, \tag{16.2}$$

where only the non-Abelian part of (14.3) is carried in (16.2).

The derivative term in (16.2) is, after the knot symbol is dropped but understood on the right-hand side, in the following equations,

$$\begin{aligned}
 (\bar{\nu} \quad \bar{l})_{\hat{L}} \partial \begin{pmatrix} \nu \\ l \end{pmatrix}_{\hat{L}} &= (\bar{c}^3 \bar{\nu}_L) \partial (c^3 \nu_L) + (\bar{a}^3 \bar{l}_L) \partial (a^3 l_L) \\
 &= (\bar{c}^3 c^3) \bar{\nu}_L \partial \nu_L + (\bar{a}^3 a^3) \bar{l}_L \partial l_L.
 \end{aligned} \tag{16.3}$$

Here $\begin{pmatrix} \nu \\ l \end{pmatrix}_{\hat{L}} \equiv \begin{pmatrix} c^3 \nu_{\hat{L}} \\ a^3 l_{\hat{L}} \end{pmatrix}$ is the knot doublet and $\begin{pmatrix} \nu \\ l \end{pmatrix}_L$ is the doublet of the standard model.

By (16.3) one has

$$(\bar{\nu} \quad \bar{l})_{\hat{L}} \partial \begin{pmatrix} \nu \\ l \end{pmatrix}_{\hat{L}} = \bar{\nu}_L \Delta_\nu \nu_L + \bar{l}_L \Delta_l l_L, \tag{16.4}$$

where

$$\Delta_\nu = \bar{c}^3 c^3 \partial, \quad \Delta_l = \bar{a}^3 a^3 \partial. \quad (16.5)$$

Then Δ_ν and Δ_l are momentum operators rescaled with the same factors that rescale the neutrino and charged lepton rest masses found in Eq. (13.17).

The second term of (16.2) is, ignoring the coupling factor, ig , and systematically dropping the understood knot symbol on the right-hand side of these equations

$$\begin{aligned} (\bar{\nu} \quad \bar{l})_{\hat{L}} \hat{\mathcal{W}} \begin{pmatrix} \nu \\ l \end{pmatrix}_{\hat{L}} &= [\bar{c}^3 \bar{\nu}_L \quad \bar{a}^3 \bar{l}_L] \begin{bmatrix} c_3 \mathcal{D}_3 W^3 & c_+ \mathcal{D}_+ W^+ \\ c_- \mathcal{D}_- W^- & -c_3 \mathcal{D}_3 W^3 \end{bmatrix} \begin{bmatrix} c^3 \nu_L \\ a^3 l_L \end{bmatrix} \\ &= [\bar{c}^3 \bar{\nu}_L \quad \bar{a}^3 \bar{l}_L] \begin{bmatrix} c_3 \mathcal{D}_3 W^3 \cdot c^3 \nu_L + c_+ \mathcal{D}_+ W^+ \cdot a^3 l_L \\ c_- \mathcal{D}_- W^- \cdot c^3 \nu_L - c_3 \mathcal{D}_3 W^3 \cdot a^3 l_L \end{bmatrix} \end{aligned} \quad (16.6)$$

$$\begin{aligned} &= c_3 (\bar{c}^3 \mathcal{D}_3 c^3) (\bar{\nu}_L W^3 \nu_L) + c_+ (\bar{c}^3 \mathcal{D}_+ a^3) (\bar{\nu}_L W^+ l_L) \\ &\quad + c_- (\bar{a}^3 \mathcal{D}_- c^3) (\bar{l}_L W^- \nu_L) - c_3 (\bar{a}^3 \mathcal{D}_3 a^3) (\bar{l}_L W^3 l_L). \end{aligned} \quad (16.7)$$

There are four form factors stemming from the knot degrees of freedom, namely:

$$F_{\bar{\nu}\nu} = c_3 \bar{c}^3 \mathcal{D}_3 c^3 = c_3 \bar{c}^3 f_3(bc) c^3, \quad (16.8)$$

$$F_{\bar{l}l} = c_3 \bar{a}^3 \mathcal{D}_3 a^3 = c_3 \bar{a}^3 f_3(bc) a^3, \quad (16.9)$$

$$F_{\bar{\nu}l} = c_+ \bar{c}^3 \mathcal{D}_+ a^3 = c_+ \bar{c}^3 (c^3 d^3) a^3, \quad (16.10)$$

$$F_{\bar{l}\nu} = c_- \bar{a}^3 \mathcal{D}_- c^3 = c_- \bar{a}^3 (a^3 b^3) c^3. \quad (16.11)$$

Here $f_3(bc) = D_{00}^3$ as in (15.11).

Then the interaction is

$$F_{\bar{\nu}\nu} (\bar{\nu}_L W^3 \nu_L) - F_{\bar{l}l} (\bar{l}_L W^3 l_L) + F_{\bar{\nu}l} (\bar{\nu}_L W^+ l_L) + F_{\bar{l}\nu} (\bar{l}_L W^- \nu_L). \quad (16.12)$$

In (16.8)–(16.11) the form factors are numerically valued on the state $|n\rangle$ fixing the Lagrangian. All of these form factors are invariant under $U_a(1) \times U_b(1)$ since a and d , as well as b and c , transform oppositely and each operator transforms oppositely to its adjoint.

For the up–down quark doublet we have

$$\bar{L} \nabla L = (\bar{u} \quad \bar{d})_{\hat{L}} (\partial + ig \hat{\mathcal{W}}) \begin{pmatrix} u \\ d \end{pmatrix}_{\hat{L}}, \quad (16.13)$$

where, in this section, d is notation on the left side of (16.14) for down quark as well as for the d preon on the right side of (16.14) as follows:

$$\begin{pmatrix} u \\ d \end{pmatrix}_{\hat{L}} = \begin{pmatrix} cd^2 \cdot u_L \\ ab^2 \cdot d_L \end{pmatrix}. \quad (16.14)$$

Here again $\begin{pmatrix} u \\ d \end{pmatrix}_{\hat{L}}$ in (16.14) is the knot doublet while $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ is the doublet in the standard model.

The derivative term of (16.13) is

$$\begin{aligned}
 (\bar{u} \quad \bar{d})_{\hat{L}} \partial \begin{pmatrix} u \\ d \end{pmatrix}_{\hat{L}} &= \overline{cd^2} \bar{u}_L \partial (cd^2) u_L + \overline{ab^2} \bar{d}_L \partial (ab^2) d_L \\
 &= (\overline{cd^2} cd^2) \bar{u}_L \partial u_L + (\overline{ab^2} ab^2) \bar{d}_L \partial d_L.
 \end{aligned} \tag{16.15a}$$

Equation (16.15a) may be rewritten as

$$(\bar{u} \quad \bar{d})_{\hat{L}} \partial \begin{pmatrix} u \\ d \end{pmatrix}_{\hat{L}} = \bar{u}_L \Delta_u u_L + \bar{d}_L \Delta_d d_L, \tag{16.15b}$$

where

$$\Delta_u = \overline{cd^2} cd^2 \partial, \quad \Delta_d = \overline{ab^2} ab^2 \partial. \tag{16.15c}$$

Here Δ_u and Δ_d are again momentum operators rescaled with the same factors that rescale the rest masses of the u and d quarks in (13.17).

The interaction term in Eq. (16.13), after dropping the knot symbol, is

$$\begin{aligned}
 c_3 [(\overline{cd^2}) \mathcal{D}_3(cd^2)] (\bar{u}_L W^3 u_L) + c_+ [(\overline{cd^2}) \mathcal{D}_+(ab^2)] (\bar{u}_L W^+ d_L) \\
 + c_- [(\overline{ab^2}) \mathcal{D}_-(cd^2)] (\bar{d}_L W^- u_L) - c_3 [(\overline{ab^2}) \mathcal{D}_3(ab^2)] (\bar{d}_L W^3 d_L).
 \end{aligned} \tag{16.16}$$

The interaction term is then the sum of four parts:

$$F_{\bar{u}u} (\bar{u}_L W^3 u_L) - F_{\bar{d}d} (\bar{d}_L W^3 d_L) + F_{\bar{u}d} (\bar{u}_L W^+ d_L) + F_{\bar{d}u} (\bar{d}_L W^- u_L), \tag{16.17}$$

where the four form factors are

$$F_{\bar{u}u} = c_3 \overline{cd^2} f_3(bc) cd^2, \tag{16.18}$$

$$F_{\bar{d}d} = c_3 \overline{ab^2} f_3(bc) ab^2, \tag{16.19}$$

$$F_{\bar{u}d} = c_+ (\overline{cd^2}) \mathcal{D}_+(ab^2) = c_+ \overline{cd^2} (c^3 d^3) ab^2, \tag{16.20}$$

$$F_{\bar{d}u} = c_- (\overline{ab^2}) \mathcal{D}_-(cd^2) = c_- \overline{ab^2} (a^3 b^3) cd^2. \tag{16.21}$$

All of these form factors are invariant under $U_a(1) \times U_b(1)$ since a and d transform oppositely as do b and c .

If one passes to $SUq(2)$ all of the four form factors may be evaluated in terms of q and β , where β is the eigenvalue of b on the ground state.

Since the R -fields are $SU(2)$ singlets, they are invariant under $SU(2)$ transformations and are not subject to $SU(2)$ interactions. They do transform according to hypercharge (t_0), or rotation charge. These are $U(1)$ gauge transformations, and $\bar{R} \nabla R$ is the sum of the following four parts:

$$\bar{c}^3 c^3 (\bar{\nu}_R (\partial + W^0) \nu_R), \tag{16.22}$$

$$\bar{a}^3 a^3 (\bar{l}_R (\partial + W^0) l_R), \tag{16.23}$$

$$\overline{cd^2} cd^2(\bar{u}_R(\partial + W^0)u_R), \tag{16.24}$$

$$\overline{ab^2} ab^2(\bar{d}_R(\partial + W^0)d_R). \tag{16.25}$$

All these terms are again invariant under $U_a(1) \times U_b(1)$ gauge transformations on the $SLq(2)$ algebra.

To obtain the corresponding terms of the preon Lagrangian replace $(\nu, l, u, d)_R$ by $(c, a, d, b)_R$ and $(\bar{c}^3 c^3, \bar{a}^3 a^3, \overline{cd^2} cd^2, \overline{ab^2} ab^2)$ by $(\bar{c}c, \bar{a}a, \bar{d}d, \bar{b}b)$, respectively.

17. The Higgs Kinetic Energy Terms

The weak neutral couplings are by (14.3)

$$i(gW_3\tau_3 + g_0W_0\tau_0) = i(\mathcal{A}A + \mathcal{Z}Z), \tag{17.1}$$

where the (t_3, t_0) of the standard model has been replaced by (τ_3, τ_0) in (14.4) and (14.5) and

$$W_0 = A \cos \theta - Z \sin \theta, \tag{17.2}$$

$$W_3 = A \sin \theta + Z \cos \theta. \tag{17.3}$$

Here θ is the Weinberg angle:

$$\tan \theta = \frac{g_0}{g}. \tag{17.4}$$

Then

$$\mathcal{A} = g_0(\tau_3 + \tau_0) \cos \theta, \tag{17.5}$$

$$\mathcal{Z} = g(\tau_3 - \tau_0 \tan^2 \theta) \cos \theta. \tag{17.6}$$

If $|0\rangle$ is any state satisfying

$$\mathcal{A}|0\rangle = 0 \tag{17.7}$$

or by (17.5)

$$(\tau_3 + \tau_0)|0\rangle = 0 \tag{17.8}$$

then by (17.6)

$$\mathcal{Z}|0\rangle = \left(\frac{\tau_3}{\cos \theta} \right) |0\rangle. \tag{17.9}$$

In the standard model and also as we shall assume in the knot model, the Higgs is coupled to the observed electroweak field. In the preon model we shall assume that the Higgs is also coupled to the hypothetical electroweak field described by the adjoint representation of $SLq(2)$. Then there are two realizations of the covariant derivative of the Higgs as follows:

$$\nabla^j \varphi|0\rangle = \partial \varphi|0\rangle + ig \left[(W^+ \tau_+)^j + (W^- \tau_-)^j + \frac{(Z \tau_0)^j}{\cos \theta} \right] \varphi|0\rangle, \quad j = 1, 3, \tag{17.10}$$

where j refers to the representation and θ is the Weinberg angle. The (W^1, W^3) implied by (17.10) were previously denoted by (w, W) in Tables 6 and 7, respectively. The corresponding contributions to the total kinetic energy are

$$\begin{aligned} & \frac{1}{2} \langle 0 | \text{Tr} \overline{\nabla_{\mu} \varphi} \nabla^{\mu} \varphi | 0 \rangle^j \\ &= \frac{1}{2} \partial_{\mu} \rho \partial^{\mu} \rho + g^2 \rho^2 \left[I_{++}^j (W_{+}^{\mu} W_{+\mu})^j + I_{--}^j (W_{-}^{\mu} W_{-\mu})^j + \frac{I_{33}^j}{\cos^2 \theta} (Z^{\mu} Z_{\mu})^j \right], \quad j = 1, 3. \end{aligned} \tag{17.11}$$

Here the charged component of φ has been transformed away and

$$\varphi = \begin{pmatrix} 0 \\ \rho \end{pmatrix},$$

where ρ is real.

Here

$$I_{kk}^j = \frac{1}{2} \text{Tr} \langle 0 | \bar{\tau}_k^j \tau_k^j | 0 \rangle, \quad \begin{aligned} k &= +, -, 3, \\ j &= 1, 3, \end{aligned}$$

where the τ_k^j are given by (14.4) and (14.5).

To agree with the observed masses of the W_+ , W_- and Z of the standard theory we have set $\langle 0 | 0 \rangle = 1$ and

$$I_{kk}^3 = 1. \tag{17.12}$$

As the simplest generalization of (17.12) we shall now set

$$I_{kk}^j = 1, \quad j = 1, 3. \tag{17.13}$$

Then the c_k^j introduced in (14.4) and (14.5) are determined as follows:

$$\boxed{|c_k^j|^{-2} = \frac{1}{2} \langle 0 | \bar{\mathcal{D}}_k^j \mathcal{D}_k^j | 0 \rangle, \quad \begin{aligned} k &= +, -, 3, \\ j &= 1, 3 \end{aligned}}. \tag{17.14}$$

The expressions $\bar{\mathcal{D}}_k^j \mathcal{D}_k^j$ are invariant under $U(a) \times U(b)$ and are therefore functions of bc . The explicit expressions for $(\mathcal{D}_+^j, \mathcal{D}_-^j, \mathcal{D}_3^j)$ are $(c^3 d^3, a^3 b^3, \mathcal{D}_{00}^3)$ for $j = 3$, and $(cd, ab, \mathcal{D}_{00}^1)$ for $j = 1$. The $|c_k^j|$ are then determined as functions of $\beta\gamma$.

18. The Field Invariant

We replace the field invariant of the standard model by

$$\langle 0 | \text{Tr} \mathcal{W}_{\mu\lambda}^j \mathcal{W}^{j\mu\lambda} | 0 \rangle, \quad j = 1, 3, \tag{18.1}$$

where $\mathcal{W}_{\mu\lambda}^j$ are the field strengths of the knot and preon models and where $|0\rangle$ is the ground state of the commuting b and c operators. In (18.1) we have

$$\mathcal{W}_{\mu\lambda}^j = [\nabla_{\mu}^j, \nabla_{\lambda}^j], \quad j = 1, 3, \tag{18.2}$$

where

$$\nabla_\mu^j = \partial_\mu + \mathcal{W}_\mu^j \tag{18.3}$$

and by (14.3)

$$\mathcal{W}_\mu^j = ig \left(W_\mu^+ \tau_+^j + W_\mu^- \tau_-^j + W_\mu^3 \tau_3^j \right) + ig_0 W^0 t_0, \quad j = 1, 3. \tag{18.4}$$

Then the non-Abelian part of the field is

$$\mathcal{W}_{\mu\lambda}^j = ig \left(\partial_\mu W_\lambda^k - \partial_\lambda W_\mu^k \right) \tau_k^j - g^2 W_\mu^k W_\lambda^\ell [\tau_k^j, \tau_\ell^j], \quad j = 1, 3, \tag{18.5}$$

where $(k, \ell) = (+, -, 0)$ and differs from the standard model by the substitution of τ_k^j for t_k .

The τ -commutators introduce structure coefficients invariant under the gauge transformations $U_a(1) \times U_b(1)$ that leave the $SLq(2)$ algebra invariant and hence are functions of bc only. The structure coefficients implied by (18.5) will therefore be functions of $\beta\gamma$, the value of bc on the ground state that appears in (18.1).

As already noted, the description of the knotted standard model in Secs. 12–18 has been simplified by not endowing the preons with gluon indices.

19. Masses and Interactions^{18,19}

The leptons and quarks are described by the $D^{3/2}$ representation and interact by the D^3 vectors, while the preons are described by the $D^{1/2}$ representation and interact by the D^1 vectors.

Since the number of preons equals the number of crossings by (11.6), one may speculate that the crossings and preons are pointlike, that there is one preon at each crossing, and that the leptons and quarks are composed of three preons bound by a trefoil of knotted electroweak and gluon fields. If this is a realistic picture, there should be three bound states, corresponding to the three generations of each family, with their observed masses; and assuming that the preon dynamics is entirely determined by the knotted action, the calculation of these bound states could be formulated as a well-defined mathematical problem. On the other hand, to reach a physically credible picture, one needs some experimental guidance at relevant and presumably very high energies. For example, one might expect the electroproduction of a and d preons according to

$$e^+ + e^- \rightarrow a + d + \dots$$

since a and d are oppositely charged $(\pm \frac{e}{3})$.

The following decay modes are also kinematically possible:

$$\begin{aligned} \text{Down quarks:} \quad & D_{\frac{3}{2}-\frac{1}{2}}^{3/2} \rightarrow D_{\frac{1}{2}\frac{1}{2}}^{1/2} + D_{1-1}^1 \quad (ab^2 \rightarrow a + b^2), \\ \text{Up quarks:} \quad & D_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \rightarrow D_{-\frac{1}{2}\frac{1}{2}}^{1/2} + D_{-1-1}^1 \quad (cd^2 \rightarrow c + d^2). \end{aligned}$$

Table 8.

i	$D_{mm'}^{3/2}(i)$	$M(i, n)$
l	a^3	$\rho(l)\langle n \bar{a}^3 a^3 n\rangle$
ν	c^3	$\rho(\nu)\langle n \bar{c}^3 c^3 n\rangle$
d	ab^2	$\rho(d)\langle n \bar{b}^2 \bar{a} \cdot ab^2 n\rangle$
u	cd^2	$\rho(u)\langle n \bar{d}^2 \bar{c} \cdot cd^2 n\rangle$

These decays could limit to three the number of generations by permitting the quark to decay from higher excited states. In that case one would expect the formation of a preon–quark plasma at a sufficiently high temperature.

Currently there is data on electroweak reaction rates and on the masses of the three generations. This data at present constrains and in principle is predicted by the knot model. To discuss this data we now introduce some simplifications based on the same physical picture and on $SUq(2)$, the unitary version of $SLq(2)$. Let us first consider the masses of the three generations of fermions.

20. The Masses of the Fermions^{18,19}

The mass terms (13.17) of the knot Lagrangian contain the mass spectra of the four families that are listed in Table 8.

These masses are all of the form

$$\rho(m, m')\langle n|\bar{D}_{mm'}^{3/2} D_{mm'}^{3/2}|n\rangle,$$

where (m, m') labels the family and n labels the generation. The $|n\rangle$ are eigenstates of $\bar{D}_{mm'}^{3/2} D_{mm'}^{3/2}$. Since the electric charge is $-\frac{e}{3}(m+m')$, the pair (m, m') determines both the mass and the charge.

In this table, as before, only the operator factor of the monomial $D_{mm'}^{3/2}$ is recorded. The four prefactors $(\rho(l), \rho(\nu), \rho(d), \rho(u))$ represent the products of the numerical factors in $D_{mm'}^{3/2}$ with the Higgs factor. The magnitude of ρ sets the energy scale and differs for each family.

The $M(i, n)$ in Table 8 are invariant under $U_a(1) \times U_b(1)$ transformations since the preon operators (a, b, c, d) transform oppositely to their adjoints.

To numerically evaluate the expectation values of these operator products we may go to the unitary version of $SLq(2)$ by setting

$$d = \bar{a}, \tag{20.1}$$

$$c = -q_1 \bar{b}. \tag{20.2}$$

Then

$$\begin{aligned} ab &= qba, & a\bar{a} + b\bar{b} &= 1, \\ a\bar{b} &= \bar{q}ba, & \bar{a}a + q_1^2 \bar{b}b &= 1, & b\bar{b} &= \bar{b}b. \end{aligned} \tag{20.3}$$

The identification of d with \bar{a} and c with \bar{b} is in agreement with the physical identification of the creation operators for the d and c preons with the creation operators, \bar{a} and \bar{b} , for the antiparticles of the a and b preons respectively. Then the operators $\bar{a}^n a^n$ and $a^n \bar{a}^n$ are charge neutral and are expressible in terms of $b\bar{b}$ which is also charge neutral.

The reduction of $a^n d^n$ to a polynomial in bc may be shown as follows:

$$\begin{aligned} a^n d^n &= a^{n-1} \cdot ad \cdot d^{n-1} \\ &= a^{n-1} (1 + qbc) d^{n-1} \end{aligned} \tag{20.4}$$

$$= a^{n-1} d^{n-1} (1 + q^{2n-1} bc). \tag{20.5}$$

By iteration one finds

$$a^n d^n = \prod_{s=1}^{2n-1} (1 + q^s bc) \tag{20.6}$$

and in $SU_q(2)$

$$a^n \bar{a}^n = \prod_{s=1}^{2n-1} (1 - q^{s-1} b\bar{b}) \tag{20.7}$$

and

$$\bar{a}^n a^n = \prod_{s=1}^n (1 - q_1^{2s} b\bar{b}). \tag{20.8}$$

We take the states $|n\rangle$ in Table 8 to be eigenstates of a mass operator expressed as a function of $b\bar{b}$. Then the expectation values for these states are functions of $\beta\bar{\beta}$, the eigenvalue of $b\bar{b}$ on the ground state. The $M(i, n)$ are then functions of (q, β, n) and ρ , but the ratios of the masses in a single family depend only on (q, β, n) and not on ρ .^{19,20}

The three generations, corresponding to the ground and two excited states, may be labeled by three choices of n . The three expressions for the mass $M(i, n)$ correspond to the three choices of n within a single family and are functions of the four parameters of the model (q, β, n, ρ) according to

$$M(i, n) = \rho(i) F(q^2, |\beta|^2, n), \quad i = (l, \nu, d, u), \tag{20.9}$$

where $F(q^2, |\beta|^2, n)$ is a polynomial in $|\beta|^2$ of the third degree, and a polynomial in q^2 of the degree determined by Table 8, Eq. (20.8), and the algebra (A).

Depending on the assignment of n to the three generations, one may determine q and β by Eq. (20.9) from the two ratios of the three observed masses.^{19,20} All of these estimates of mass ratios are based on the electroweak Lagrangian and need to be corrected to include the effects of gluon charge.

To obtain the absolute masses one needs to know the $\rho(i)$. The wide spread in the observed masses then requires a wide spectrum of $\rho(i)$ and suggests a family of Higgs particles also with widely different masses, or a single Higgs with widely different couplings.

21. Electroweak Reaction Rates

The matrix elements of the standard model acquire the following form factors in the corresponding knot model

$$\langle n'' | \bar{D}_{-3t_3''-3t_0''}^{3/2} D_{-3t_3'-3t_0'}^3 D_{-3t_3-3t_0}^{3/2} | n \rangle, \tag{21.1}$$

where n and n'' run over the three generations.

As an example consider

$$l^- + W^+ \rightarrow \nu_l \tag{21.2}$$

with the following form factor

$$\langle \nu_l | \bar{D}_{-\frac{3}{2}\frac{3}{2}}^{3/2} D_{-30}^3 D_{\frac{3}{2}\frac{3}{2}}^{3/2} | l \rangle. \tag{21.3}$$

If this form factor is reduced in the $SUq(2)$ algebra, it becomes a function of q and β , where β is the eigenvalue of b on the ground state $|0\rangle$. Comparison of (21.3) with experimental data on lepton–neutrino interactions like (21.2) indicates that

$$(q, \beta) \cong \left(1, \frac{\sqrt{2}}{2} \right) \tag{21.4}$$

in approximate agreement with the universal Fermi interaction.²⁰

More demanding tests of the model are provided by the CKM and PMNS matrices that relate to (21.1), depending on whether n and n'' refer to quarks or charged leptons and neutrinos. In making these tests we examine the possibility that the flavor states are simply related to the “coherent states,” i.e. the eigenstates of the operators \bar{a} and a , that are raising and lowering operators and thereby transmute particles of one generation into particles of the adjoining generation.^{15,21}

Starting from the mass states, one may obtain the coherent states as follows. The orthonormal mass states $|n\rangle$ are defined to satisfy

$$|n\rangle \sim \bar{a}^n |0\rangle, \tag{21.5}$$

$$\langle n | n' \rangle = \delta(n, n'). \tag{21.6}$$

Then \bar{a} is a raising operator:

$$\bar{a}|n\rangle = \lambda_n |n + 1\rangle \tag{21.7}$$

and

$$\langle n | a = \lambda_n^* \langle n + 1|. \tag{21.8}$$

By (21.6) and (20.3)

$$|\lambda_n|^2 = 1 - q^{2n} |\beta|^2. \tag{21.9}$$

Similarly, a , working to the right on $|n\rangle$, is a lowering operator.

Let $|\alpha\rangle$ be an eigenstate of a with eigenvalue α :

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{21.10}$$

$$\langle\alpha|\bar{a} = \langle\alpha|\alpha^*. \tag{21.11}$$

We now compute the matrix element $\langle n|a|\alpha\rangle$ connecting mass and coherent states.

If a operates to the right, one has by (21.10)

$$\langle n|a|\alpha\rangle = \alpha\langle n|\alpha\rangle \tag{21.12}$$

and if it operates to the left, one has by (21.8)

$$\langle n|a|\alpha\rangle = \lambda_n^*\langle n+1|\alpha\rangle. \tag{21.13}$$

Then

$$\langle n+1|\alpha\rangle = \frac{\alpha}{\lambda_n^*}\langle n|\alpha\rangle. \tag{21.14}$$

By iteration,

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\prod_0^{n-1} \lambda_s^*} \langle 0|\alpha\rangle \tag{21.15}$$

and $\langle 0|\alpha\rangle$ may be fixed by normalizing $|\alpha\rangle$.

Let the flavor states $|i\rangle$ be expressed as superpositions of the mass states $|n\rangle$:

$$|i\rangle = \sum |n\rangle\langle n|i\rangle. \tag{21.16}$$

Then the matrix elements between flavor states are related to the matrix elements between mass states as follows

$$\langle i|M|i'\rangle = \sum \langle i|n\rangle\langle n|M|n'\rangle\langle n'|i'\rangle. \tag{21.17}$$

Let

$$N_i = \langle i|i\rangle. \tag{21.18}$$

We are now interested in the generalization of (21.3) to the cases for which

$$M = \bar{D}_{m''p''}^{3/2} D_{m'p'}^3 D_{mp}^{3/2}, \tag{21.19}$$

when taken between flavor states, modifies the weak vector interactions of all the elementary fermions. In particular

$$\langle u(i)|W^+|d(i')\rangle = \sum_{nn'} \langle u(i)|u(n)\rangle\langle u(n)|W^+|d(n')\rangle\langle d(n')|d(i')\rangle, \tag{21.20}$$

$$\langle d(i)|W^-|u(i')\rangle = \sum_{nn'} \langle d(i)|d(n)\rangle\langle d(n)|W^-|u(n')\rangle\langle u(n')|u(i')\rangle \tag{21.21}$$

holding for the flavor states of the up and down quarks.

With the same model for the PMNS matrix the form factor is

$$\langle i | \bar{D}_{-\frac{3}{2}\frac{3}{2}}^{3/2} D_{00}^3 D_{-\frac{3}{2}\frac{3}{2}}^{3/2} | i' \rangle, \quad (21.22)$$

where $i = 0, 1, 2$ label the three generations of neutrino flavor states.

Because of the $U_m(1) \times U_p(1)$ symmetry, the matrix element M in (21.19) is neutral, i.e. $n_a - n_d = n_b - n_c = 0$. It is therefore a function of b and c only and has no off-diagonal elements:

$$\langle n | M | n' \rangle = M_n \delta(n, n') \quad (21.23)$$

and by (21.17)

$$\langle i | M | i' \rangle = \sum_n \langle i | n \rangle M_n \langle n | i' \rangle. \quad (21.24)$$

If the empirical flavor states $|i\rangle$ are tentatively identified as coherent states $|\alpha\rangle$, then $\langle i | M | i' \rangle$ is the mixing matrix and may be parametrized by $\langle n | i \rangle$.

The quantities $|\langle n | i \rangle|$ may be expressed as a function of

- (a) the eigenvalues of a : α
- (b) the norms of the eigenstates of a : N_i
- (c) the matrix elements of a between neighboring mass states: $\lambda_n^* = \langle n | a | n + 1 \rangle$

and the electroweak mixing matrix may be parametrized by α , N_i and λ_n .

The CKM and PMNS mixing matrices are 3×3 empirical matrices. The CKM matrix has been expressed by Kobayashi and Maskawa as a unitary matrix with three mixing angles and a CP-violating KM phase. The CKM empirical matrix may also be parametrized with $(\alpha, N_i, \lambda_n, q)$ of the knot model, but these parameters must be related since only 4 parameters are needed as shown by Wolfenstein and by Kobayashi and Maskawa. Hence, if the flavor states are tentatively defined by or closely related to coherent states, the empirical CKM matrix puts only weak restrictions on $(\alpha, N_i, \lambda_n, q)$.

22. Effective Hamiltonians for Composite Leptons and Quarks

Although the masses and interactions of the composite charged leptons, neutrinos and quarks can be expressed in terms of the knot parameters, q , β , γ , it may also be possible to obtain an approximate description of these 12 particles as the three-preon structures that are schematically pictured in Fig. 3.

The $SU(3)$ charge structure of the up and down quarks resembles the electric charge structure of the H^3 nucleus, i.e. one charged (a_i or c_i) and two neutral (b or d) particles, resembling one proton and two neutrons.

The three preons in these hypothetical structures are bonded by the hypothetical preonic vector field $D_{mm'}^1$ with the operator form factors shown in Table 9.

It is in principle possible to obtain the eigenvalues of the Hamiltonian that corresponds to the Lagrangian of the knot model. Instead of attempting to solve this difficult problem, however, one may try to gain an approximate picture of

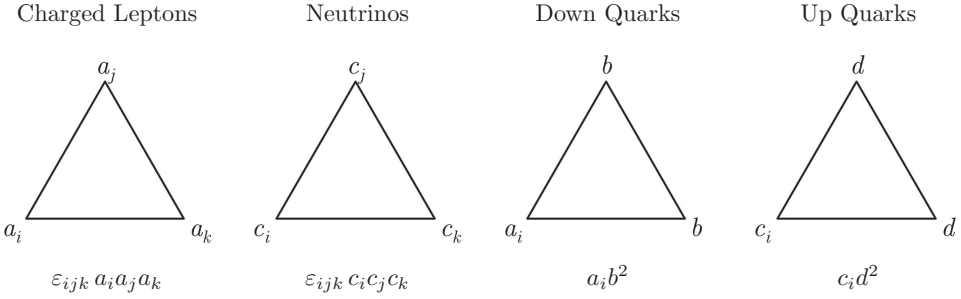


Fig. 3. Schematic representations of elementary fermions.

Table 9.

	$D_{mm'}^{3/2}$	Bond	Form factor
charged leptons	a^3	$a - a$	$\bar{D}_{\frac{1}{2}\frac{1}{2}}^{1/2} D_{00}^1 D_{\frac{1}{2}\frac{1}{2}}^{1/2} = \bar{a}(ad + bc)a$
neutrinos	c^3	$c - c$	$\bar{D}_{-\frac{1}{2}\frac{1}{2}}^{1/2} D_{00}^1 D_{-\frac{1}{2}\frac{1}{2}}^{1/2} = \bar{c}(ad + bc)c$
down quarks	ab^2	$a - b$	$\bar{D}_{\frac{1}{2}\frac{1}{2}}^{1/2} D_{01}^1 D_{\frac{1}{2}-\frac{1}{2}}^{1/2} = \bar{a}(ac)b$
		$b - b$	$\bar{D}_{\frac{1}{2}-\frac{1}{2}}^{1/2} D_{00}^1 D_{\frac{1}{2}-\frac{1}{2}}^{1/2} = \bar{b}(ad + bc)b$
up quarks	cd^2	$c - d$	$\bar{D}_{-\frac{1}{2}\frac{1}{2}}^{1/2} D_{01}^1 D_{-\frac{1}{2}-\frac{1}{2}}^{1/2} = \bar{c}(ac)d$
		$d - d$	$\bar{D}_{-\frac{1}{2}-\frac{1}{2}}^{1/2} D_{00}^1 D_{-\frac{1}{2}-\frac{1}{2}}^{1/2} = \bar{d}(ad + bc)d$

these three-particle composite particles with the aid of effective Hamiltonians for three-particle systems.

The topological diagrams in Fig. 2 may be shrunk into the effectively triangular shapes in Fig. 3 by reducing the outside loops of the trefoils to infinitesimal loops. Then the two body forces result from the matrix elements that connect the fermionic preons and are mediated by the preonic vectors. The relevant two body bonds are

$$\text{charged leptons: } a - a, \quad \text{neutrinos: } c - c,$$

$$\text{down quarks: } a - b \text{ and } b - b, \quad \text{up quarks: } c - d \text{ and } d - d.$$

Then we have Table 9 describing operator form factors for the two body forces.

The operator form factors may all be reduced by the algebra (A) to functions of bc and q . If one then reduces these operators by replacing $SL_q(2)$ with $SU_q(2)$, one finds that the strength of these bonds depends on the values of q and $\bar{b}b$.

Within the $SL_q(2)$ kinematics there are several options in constructing an effective Hamiltonian for the three body structures that represent the charged leptons, neutrinos, and quarks. These possible three body effective Hamiltonians permit

gluon and gravitational forces, but the model, as here described, allows only electro-weak interactions that are mediated by the preonic adjoint field and are therefore proportional to the form factors in Table 9. A successful model must have an angular momentum of $\frac{h}{2}$ and should also have only three bound states, corresponding to the three generations. The quanta of the binding field lie in the adjoint representation and are represented by normal modes of $w_\mu^+(x)D_{-10}^1$, $w_\mu^-(x)D_{10}^1$, $z_\mu(x)D_{00}^1$ and $a_\mu(x)D_{00}^1$, where $w_\mu^\pm(x)$, $z_\mu(x)$ and $a_\mu(x)$ are determined by the standard model Lagrangian as modified by form factors stemming from D_{m0}^1 .

The preonic photon field $a_\mu D_{00}^1$ produces a Coulombic potential while the charged w particles and the neutral z , being massive, are responsible for a Yukawa-type potential. The range of the Yukawa potentials is determined by the masses of the charged w and the neutral z . The ‘‘Higgs masses’’ of these particles are in turn determined by the vacuum expectation values of the Higgs scalars and are rescaled by

$$\langle 0 | \bar{D}_{mm'}^j D_{mm'}^j | 0 \rangle, \tag{22.1}$$

where $j = 1$ for these preonic vector fields.

All of these masses are dependent on q , β and γ . Similarly the form factors, rescaling the interactions, are dependent on the same parameters. One can construct a formal effective Hamiltonian dependent on the parameters q , β and γ .

In order to achieve an adequately strong binding at very short range, it is important that the parameters (q, β, γ) , determining the strength of the form factors, and the factor coming from the Higgs scalar, determining the range of the Yukawa potentials, be themselves sufficiently large or small. The magnitude of these parameters in turn depends on their possible physical meaning, which we now briefly consider.

23. A Speculative Interpretation of the $SLq(2)$ Algebra and of the Deformation Parameter q ²²

In Sec. 3, an implicit connection between the $SLq(2)$ algebra and the 2d projections of the classical 3d-knots was made through the matrix

$$\varepsilon_q = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix}, \tag{23.1}$$

which is invariant under the following transformation

$$T \varepsilon_q T^t = T^t \varepsilon_q T = \varepsilon_q, \tag{23.2}$$

where the elements of T define the $SLq(2)$ algebra and where ε_q underlies the Kauffman algorithm for associating the Kauffman polynomial with a knot.

If ε_q is replaced by

$$\varepsilon = \begin{pmatrix} 0 & \alpha_2 \\ -\alpha_1 & 0 \end{pmatrix}, \tag{23.3}$$

then the SLq(2) algebra (A) is again generated by (23.2) but with

$$q = \frac{\alpha_1}{\alpha_2}. \tag{23.4}$$

If one further imposes

$$\det \varepsilon = 1 \tag{23.5}$$

then

$$\alpha_1 \alpha_2 = 1 \tag{23.6}$$

and

$$\varepsilon_q = \varepsilon. \tag{23.7}$$

Then (23.3), restricted by (23.5), is equivalent to (23.1) and the knot model may be based on either ε or ε_q . By taking advantage of the fact that ε is a two-parameter matrix while ε_q depends on only a single parameter, however, one may describe a wider class of physical theories with ε . If the physical situation that the theory is being asked to describe is characterized by two interacting gauge fields, with two charges, g and g' , on the same particle, one may attempt to give physical meaning to q by embedding g and g' in ε as follows

$$\varepsilon = \begin{pmatrix} 0 & \frac{g(E)}{\sqrt{\hbar c}} \\ -\frac{g'(E)}{\sqrt{\hbar c}} & 0 \end{pmatrix}, \tag{23.8}$$

where $g(E)$ and $g'(E)$ are energy-dependent coupling constants that have been normalized to agree with experiment at hadronic energies.

Then q is defined by (23.8) and (23.4) as

$$q(E) = \frac{g'(E)}{g(E)}. \tag{23.9}$$

If (23.5) is also imposed, then

$$g(E)g'(E) = \hbar c. \tag{23.10}$$

In the electroweak knot model it is argued that the electroweak experimental data suggest an SLq(2) extension of the standard model. To the extent that this view is correct it appears that the sources of the electroweak field may be knotted, but the possible physical origins of the additional “knot” degrees of freedom have not been identified. A possible origin of the “knotting” is the deformation of the electroweak $SU(2) \times U(1)$ structure by $SU(3)$. Since the leptons and neutrinos, appearing as a^3 and c^3 particles in the SLq(2) model, have already been given $SU(3)$ indices to protect the Pauli principle, the gluon field is implicit in this model and a possible interpretation of (23.9) is then $(g', g) = (e, g)$ or (g, e) , where g is the gluon charge, and e is an electroweak coupling constant.

If (23.5) is imposed, (23.10) would become

$$eg = \hbar c, \quad (23.11)$$

which is like the Dirac restriction on magnetic poles.

Since g and e are running coupling constants, the $SLq(2)$ parameter q , which is either $\frac{e}{g}$ or $\frac{g}{e}$, is also a running and dimensionless coupling constant. If e increases with energy and g decreases with energy according to asymptotic freedom, q may become very large or very small at the high energies where the interaction and mass terms become relevant for fixing the three particle bound states representing charged leptons, neutrinos and quarks. Although there is currently no experimental data suggesting the interpretation of q as a particular function of an e and a g , such a relation (resulting from a possible physical interpretation of the otherwise physically undefined matrix ε in (23.8)) could be explored since e , g and q can be independently measured.

24. Bound Preons

The three particle models discussed here resemble familiar composite particles like H^3 , but it is possible that the H^3 example is not appropriate and that the preons are always bound. In this case the preons may not have an independent existence but may be particular field structures, or elements of larger field structures, carrying no independent degrees of freedom. In the $SLq(2)$ model described here the elementary fermions are three-preon composite particles bound by a trefoil field structure. It is possible that the trefoil field structure is a trefoil flux tube carrying energy, momentum and charge, and that energy, momentum and charge are concentrated at the three crossings. It is then possible to regard these three concentrations of energy, momentum and charge at the three crossings as actually defining the three preons without postulating their independent existence with independent degrees of freedom. Since the number of preons in any composite particle is always equal, in the $SLq(2)$ model, to the number of crossings (by (11.6)), this alternative view of the preons as tiny solitonic regions of field surrounding the crossings holds for all composite particles considered here. This alternative view of the elementary particles as lumps of field is sometimes described as a unitary field theory and has also been examined in other solitonic contexts.

25. Gravitational Binding

A major uncertainty in these realizations of the knot model which is inherited from the standard model and not introduced by the knot model, lies in the unknown values of the Higgs factors and more fundamentally in the nature of the Higgs fields and their relation to the gravitational field. It is not possible to construct a more predictive $SLq(2)$ modification of the standard model until the Higgs factor, as well as q and β , are better understood.

Since the preons are very heavy and the leptons and quarks are very tiny, gravitational attraction should not be dismissed. Since the Higgs term determines inertial mass, the Higgs factor and Higgs field must be related to the gravitational field and to supergravity and superstring models as well as to quantized Einstein gravity. It is interesting that knot states have emerged in a natural way in attempts to quantize general relativity.²³

26. Summary and Comments on the Structure of the Model

We briefly summarize and comment on the structure of the model. First define the structure of the $SL_q(2)$ algebra with the aid of the matrices ε and T as in Sec. 23:

$$T\varepsilon T^t = T^t\varepsilon T = \varepsilon, \tag{26.1}$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \varepsilon = \begin{pmatrix} 0 & \alpha_2 \\ -\alpha_1 & 0 \end{pmatrix}. \tag{26.2}$$

Here a, b, c, d are elements of the two-dimensional representation of $SL_q(2)$:

$$\begin{aligned} ab &= qba, & bd &= qdb, & ad - qbc &= 1, & bc &= cb, \\ ac &= qca, & cd &= qdc, & da - q_1cb &= 1, & q_1 &\equiv q^{-1}, \end{aligned} \tag{26.3}$$

and

$$q = \frac{\alpha_1}{\alpha_2}.$$

The $(2j + 1)$ -dimensional representation of the $SL_q(2)$ algebra is

$$D_{mm'}^j = \sum_{\substack{\delta(n_a+n_b, n_+) \\ \delta(n_c+n_d, n_-)}} A(q, n_a, n_c) \delta(n_a + n_c, n'_+) a^{n_a} b^{n_b} c^{n_c} d^{n_d}, \tag{26.4}$$

where a, b, c, d satisfy (26.3), and (n_a, n_b, n_c, n_d) are summed over all positive integers.

The structure of (26.4) implies

$$n_a + n_b + n_c + n_d = 2j, \tag{26.5}$$

$$n_a + n_b - n_c - n_d = 2m, \tag{26.6}$$

$$n_a - n_b + n_c - n_d = 2m'. \tag{26.7}$$

The essential step in the $SL_q(2)$ extension of the Lagrangian of the standard model is the replacement or renormalization of the field operators of the standard model as follows:

$$\Psi(x) \rightarrow \hat{\Psi}_{mm'}^j(x) D_{mm'}^j. \tag{26.8}$$

The new operator Lagrangian differs from the Lagrangian of the standard model by operator form factors generated by the $D_{mm'}^j$. Then the numerically valued Lagrangian depends on the state of the knot algebra on which it is evaluated.

The renormalization described by (26.8) is formally possible for any Lagrangian, including the gluon and gravitational Lagrangians, and its physical meaning depends on the physical meaning of (j, m, m') . If the particles described by the gluon and gravitational Lagrangians also have electroweak charges, then the recipe for the $SLq(2)$ extension described here for the electroweak action also holds for the gravitational and gluon actions.

The expression (26.4) for $D_{mm'}^j$ is the general representation of the knot algebra that we are proposing to define the kinematical characterization of the quantum knot. The description by (j, m, m') of the quantum knot is related to the description by (N, w, r) of the 2d projection of the corresponding classical knot as follows:

$$(j, m, m') = \frac{1}{2}(N, w, r + o). \quad (26.9)$$

As a quantum trefoil, the quantum knot is also empirically related to the left-chiral elementary fermion of the standard model by

$$(j, m, m') = 3(t, -t_3, -t_0), \quad (26.10)$$

where $(t, -t_3, -t_0)$ are the electroweak isotopic spin quantum numbers and where $t = \frac{1}{2}$ and $j = \frac{3}{2}$. Equation (26.10) holds at the $j = \frac{3}{2}$ level of the knot model. We have examined the class of left chiral states that satisfy (26.5)–(26.10).

Here are two complementary descriptions, namely (N, w, r) and (t, t_3, t_0) expressed in (26.9) and (26.10) where the (N, w, r) are interpreted as descriptive of flux loops carrying charge-current and the (t, t_3, t_0) are quantum numbers of charged particles. In the algebraic formulation of the same two descriptions (j, m, m') is the $SLq(2)$ label and (t, t_3, t_0) is the $SU(2) \times U(1)$ label. The local isotopic-electroweak group $SU(2) \times U(1)$ describes the non-Abelian electroweak field as it appears in the standard model while the global $SLq(2)$ algebra describes the global form factors that appear in the knotted standard model. To agree with the observed and conjectured electroweak fields, both ways of assigning charge require the use of fractional charges to supplement the classification by representations of a group or of an algebra. The fractional charge of the preon is $e/3$.

It is essential that all knot form factors be invariant under general gauge transformations induced by the $U_a(1) \times U_b(1)$ gauge transformations that leave the defining algebra (26.3) invariant. In particular the form factor appearing in the Higgs mass term is the product of the three knot factors for the left chiral, the Higgs, and the right chiral fields. *We assign a special status to the Higgs as an $SLq(2)$ singlet. Then to satisfy the required invariance of the knot Lagrangian, the left and right chiral fields must carry the same knot factor.* Therefore, although the left and right chiral components are doublets and singlets, respectively, of $SU(2)$, they belong to

the same representation of $SL_q(2)$. The right chiral field carries the same $D_{mm'}^j$ factor as the left chiral field.

The $D_{mm'}^j$ as described by (26.4) are creation operators for the superpositions of composite particles composed of numbers (n_a, n_b, n_c, n_d) of (a, b, c, d) preons that are allowed by the (j, m, m') indices, which in turn describe the numbers of crossings, the writhe and the rotation of the corresponding classical knot. The precise restrictions on the (n_a, n_b, n_c, n_d) are expressed by the Diophantine equations (26.5)–(26.7).

In the knot extension of the standard model the leptons and quarks lie in the $j = \frac{3}{2}$ representation of $SL_q(2)$ and are bound by vectors lying in the $j = 3$ representation. In the associated preon model the leptons and quarks are composed of three $j = \frac{1}{2}$ preons that are bound by vectors lying in the $j = 1$ representation. The $j = 0$ states are neutral electroweak loops of field flux with $r = \pm 1$ and with mass determined entirely by the Higgs factor, which is the vacuum expectation value of the Higgs field. It is then also possible to identify the $j = 0$ states with a Higgs particle.

Equations (26.9) and (26.10) permit (26.5)–(26.7) to be rewritten as

$$(t, t_3, t_0) = \sum_p n_p(t_p, t_{3_p}, t_{0_p}), \quad p = a, b, c, d, \quad (26.11)$$

$$(N, w, \tilde{r}) = \sum_p n_p(N_p, w_p, \tilde{r}_p), \quad p = a, b, c, d. \quad (26.12)$$

Then (26.11) and (26.12) express the charge and topological properties of composite structures as additive compositions of the corresponding properties of preons. These equations are complementary descriptions of a composite structure composed of either preonic particles as in (26.11) or as preonic flux tubes as in (26.12). Since the number of preonic particles is the same as the number of crossings, these two complementary pictures of leptons and quarks may be reconciled by picturing the preonic particles at the crossings of preonic flux tubes, as discussed in Sec. 11. If the preonic particles have independent degrees of freedom, they may not be bound. If they do not have independent degrees of freedom, they may simply represent the concentrations of energy, momentum and charge at crossings of the flux tubes. In this case the preons will be bound.

The $SL_q(2)$ induced form factors appearing in Secs. 13–16 rescale the energy–momentum operators as well as the interactions of the standard model. This rescaling would have implications for high energy behavior and convergence of the model, since it is a function of a possibly highly energy dependent q , as discussed in Sec. 23. We finally note another way of looking at these modifications of behavior at high energy that is related to the following deformation of the Heisenberg algebra by $SL_q(2)$.

The quadratic form

$$K = A^t \varepsilon_q A, \quad (26.13)$$

where again

$$\varepsilon_q = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix} \quad (26.14)$$

is invariant under $SL_q(2)$ transformations of A .

Choosing

$$A = \begin{pmatrix} D_x \\ x \end{pmatrix} \quad \text{and} \quad K = q^{-\frac{1}{2}} \quad (26.15)$$

(26.13) becomes

$$D_x x - q x D_x = 1. \quad (26.16)$$

Equation (26.16) is identically satisfied if D_x is chosen as the q -difference operator, namely

$$D_x \Psi(x) = \frac{\Psi(qx) - \Psi(x)}{qx - x}. \quad (26.17)$$

If we introduce

$$P_x = \frac{\hbar}{i} D_x \quad (26.18)$$

then one has the $SL_q(2)$ invariant relation

$$(P_x x - q x P_x) \Psi = \frac{\hbar}{i} \Psi. \quad (26.19)$$

If $q = 1$ then (26.19) becomes the standard Heisenberg (p, x) -commutator applied to a quantum state; but here $q \neq 1$ and D and P are difference operators given by (26.17) where q may play the role of a dimensionless regulator in field theoretic calculations, again with implications for high energy behavior and convergence of any $SL_q(2)$ model where q may be highly energy dependent.

Since the $SL_q(2)$ extended model suggests a finer level of structure than the standard model provides alone, there are clearly many open problems including questions concerning gravitational binding, the actual existence of free preons, and the renormalization of the Lagrangian dynamics.

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References

1. W. H. Thomson, *Trans. R. Soc. Edinb.* **25**, 217 (1969).
2. L. Faddeev and A. J. Niemi, *Nature* **387**, 58 (1997).
3. L. H. Kauffman, *Int. J. Mod. Phys. A* **5**, 93 (1990).
4. L. H. Kauffman, *Topology* **26**, 395 (1987).

5. N. Yu. Reshetikhin, L. A. Takhtadzhyan and L. D. Faddeev, *Leningrad Math J.* **1**, 193 (1990).
6. J. Woronowicz, *Commun. Math. Phys.* **111**, 613 (1987).
7. G. Gasper and M. Rahman, *Hypergeometric Series* (Cambridge University Press, 1990).
8. H. Weyl, *Theory of Groups and Quantum Mechanics*, ed. E. P. Dutton (Dover, 1931).
9. R. J. Finkelstein, *J. Math. Phys.* **8**, 443 (1967).
10. R. J. Finkelstein, Solitonic models based on quantum groups and the standard model, arXiv:1011.2545v3 [hep-th].
11. R. J. Finkelstein, *Int. J. Mod. Phys. A* **24**, 2307 (2009).
12. H. Harari, *Phys. Lett. B* **86**, 83 (1979).
13. M. Shupe, *Phys. Lett. B* **86**, 87 (1979).
14. J. Sonnenschein, private communication.
15. R. J. Finkelstein, *Phys. Rev. D* **89**, 125020 (2014).
16. K. Huang, *Quarks, Leptons, and Gauge Fields* (World Scientific, 1952), Eq. (6.22).
17. R. J. Finkelstein, The preon sector of the $SL_q(2)$ model, arXiv:1301.6440v3 [hep-th].
18. R. J. Finkelstein, *Int. J. Mod. Phys. A* **29**, 1450092 (2014).
19. R. J. Finkelstein, *Int. J. Mod. Phys. A* **20**, 6487 (2005).
20. A. C. Cadavid and R. J. Finkelstein, *Int. J. Mod. Phys. A* **21**, 4269 (2006).
21. R. J. Finkelstein, Flavor states of the knot model, arXiv:1011:0764 [hep-th].
22. R. J. Finkelstein, Deformation parameter in $SL_q(2)$ models of the elementary particles, arXiv:1108.0438 [hep-th].
23. G. T. Horowitz, *Strings and Symmetries* (World Scientific, 1991).
24. J. Smit, private communication.