U.S. NAVAL RESEARCH LABORATORY

Bayesian Techniques for Plasma Theory to Bridge the Gap Between Space and Lab Plasmas

Bringing Space Down to Earth
UCLA

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11 April 2017

Laboratory experiments to understand space plasmas has a long history

- 1942 - Alfvén theorises the existence of *electromagnetic-hydromagnetic* waves in a paper published in *Nature.*
  - And nobody believes him.
- 1949 – Alfvén gives a talk at U of Chicago. Fermi says "Of course such waves could exist."
  - And everybody believed him.
- 1949 – Lundquist produced Alfvén waves in magnetized mercury experiments (heavily damped).
- 1952 – Alfvén waves produced in laboratory helium plasma (Bostick).
- 1958 – Alfven waves observed on ground due to detonation of ionospheric nuclear device.
- 1960 – Alfven waves observed via magnetometer on Pioneer V

HANNES ALFVÉN

WELINDER JAEGER BERGNE

30 MAY 1908 · 2 APRIL 1995

Early work was based on linear theory

NRL PPD

Existence of Electromagnetic-Hydrodynamic Waves

If a conducting liquid is placed in a constant magnetic field, every motion of the liquid gives rise to an E.M.F. which produces electric currents. Owing to the magnetic field, these currents give mechanical forces which change the state of motion of the liquid. Thus a kind of combined electromagnetic-hydrodynamic wave is produced which, so far as I know, has as yet attracted no attention.

The phenomenon may be described by the electrodynamic equations

$$\text{rot } H = \frac{4\pi}{c} i$$

$$\text{rot } E = -\frac{1}{c}\frac{dB}{dt}$$

$$B = \mu H$$

$$i = \sigma\left(E + \frac{v}{c} \times B\right);$$

together with the hydrodynamic equation

$$\partial \frac{dv}{dt} = \frac{1}{c}(i \times B) - \text{grad } p,$$

where σ is the electric conductivity, μ the permeability, ∂ the mass density of the liquid, i the electric current, v the velocity of the liquid, and p the pressure.

Consider the simple case when $\sigma = \infty$, $\mu = 1$ and the imposed constant magnetic field H_0 is homogeneous and parallel to the z-axis. In order to study a plane wave we assume that all variables depend upon the time t and z only. If the velocity v is parallel to the x-axis, the current i is parallel to the y-axis and produces a variable magnetic field H' in the x-direction. By elementary calculation we obtain

$$\frac{d^2H'}{dz^2} = \frac{4\pi\partial}{H_0^2}\frac{d^2H'}{dt^2},$$

which means a wave in the direction of the z-axis with the velocity

$$V = \frac{H_0}{\sqrt{4\pi\partial}}.$$

Waves of this sort may be of importance in solar physics. As the sun has a general magnetic field, and as solar matter is a good conductor, the conditions for the existence of electromagnetic-hydrodynamic waves are satisfied. If in a region of the sun we have $H_0 = 15$ gauss and $\partial = 0{\cdot}005$ gm. cm.$^{-3}$, the velocity of the waves amounts to

$$V \sim 60 \text{ cm. sec.}^{-1}.$$

This is about the velocity with which the sunspot zone moves towards the equator during the sunspot cycle. The above values of H_0 and ∂ refer to a distance of about 10^{10} cm. below the solar surface where the original cause of the sunspots may be found. Thus it is possible that the sunspots are associated with a magnetic and mechanical disturbance proceeding as an electromagnetic-hydrodynamic wave.

The matter is further discussed in a paper which will appear in *Arkiv för matematik, astronomi och fysik*.

H. Alfvén.

Kgl. Tekniska Högskolan,
Stockholm.
Aug. 24.

- Linear theory is relatively easy to do.
    - Solutions are guaranteed.
- Linear theory gives good "observables".
    - Speeds
    - Dispersion relations
    - Polarizations
- Consequently, linear theory has been mostly worked out, verified in laboratory experiments, and observed in space plasmas.

A lot of nonlinear work has been done

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- Weak turbulence theory
  - Quasilinear theory
    - Observables: rates, resonance conditions
  - Decay/Coalescence (three wave) parametric
    - Observables: rates, matching conditions
  - Induced nonlinear scattering (2-wave + particle)
    - Observables: rates, resonance conditions
  - Inverse cascades and equilibrium solutions to wave kinetic equation
    - Observables: power law PSD scaling laws
- Soliton/BGK/coherent structures theory
  - Observables: Waveform properties, soliton collisions, speeds
- Strong Turbulence Theory
  - Scaling arguments lead to power law PSD scaling with k or omega

Sources **near wave effects in laboratory plasmas: A comparison between theory and experiment**

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The rich nonlinear phenomena that occur in plasmas are reviewed in a systematic way. The foundations of turbulence theory (both weak and strong) and experiments performed in the past decade to verify such theories are presented. The aim is to emphasize those experiments that demonstrate clearly the validity (or failure) of some of the theories. In particular, we discuss experiments that demonstrate the validity and/or limits of weak turbulence theory, strong turbulence theory, parametric instabilities, echoes, trapping of particles in large-amplitude waves, and electrostatic ion acoustic shocks. We present concluding remarks in each section regarding the present status of each of these phenomenon.

Rev. Mod. Physics 1978

Evidence for Nonlinear Induced Scattering of Lightning Generated Whistlers in the Radiation Belts: Supported by Laboratory Experiments

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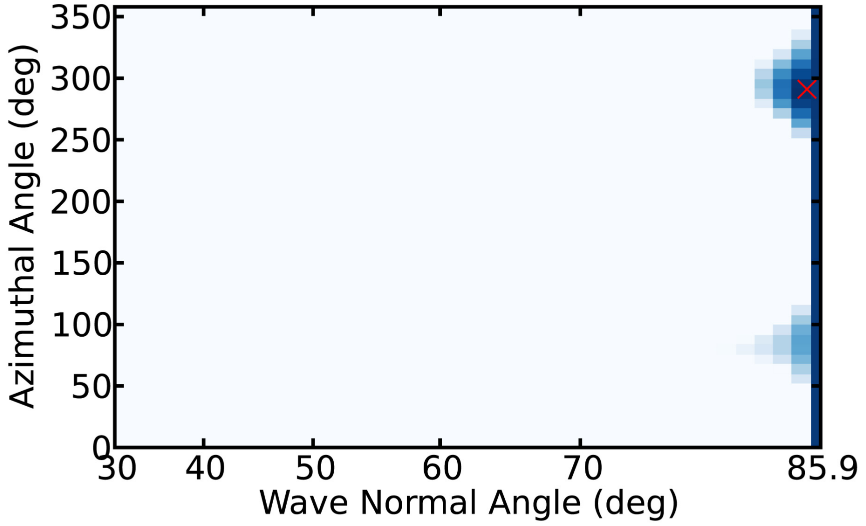
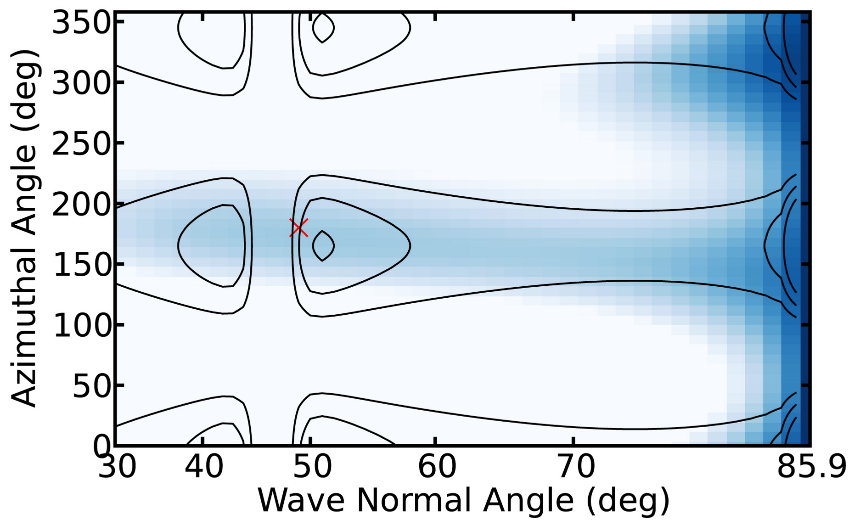
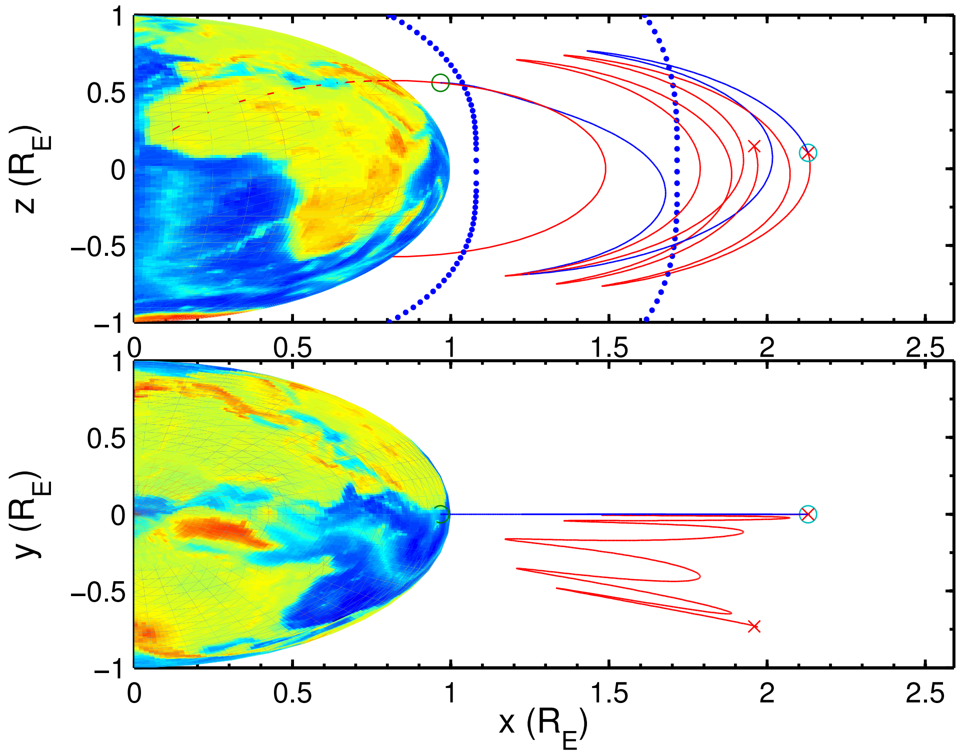
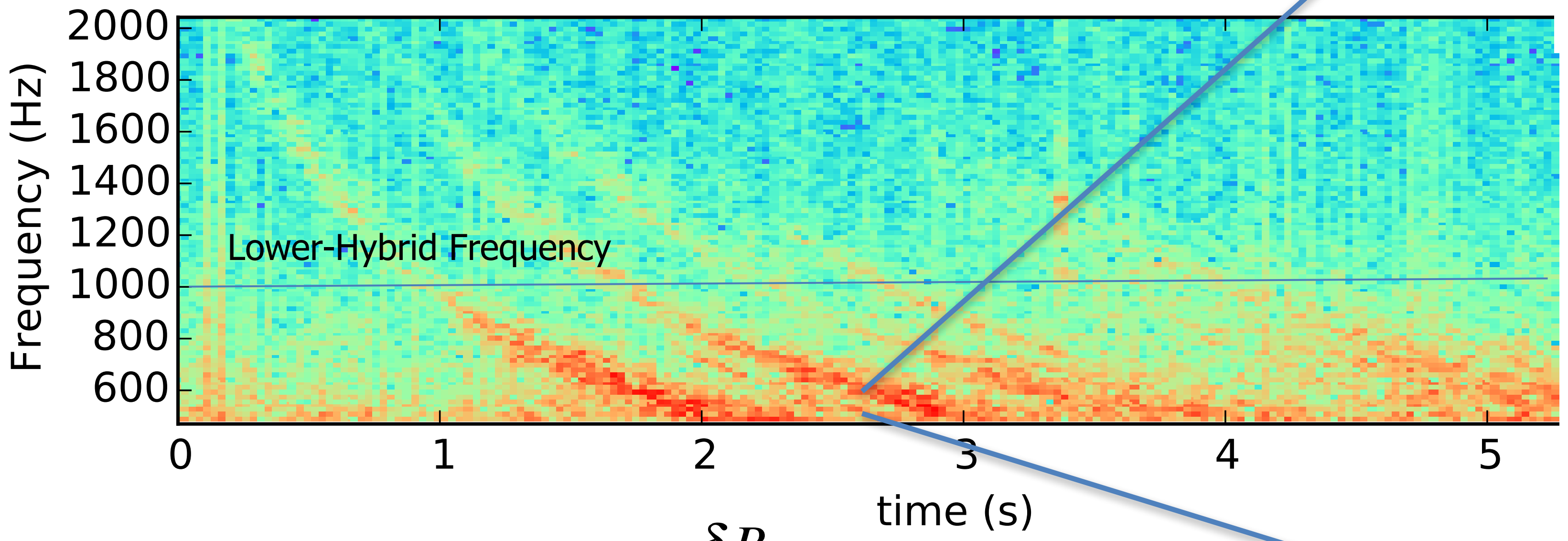
10 MHz – Lab Pump Wave



9.96 MHz – Lab Scattered Wave

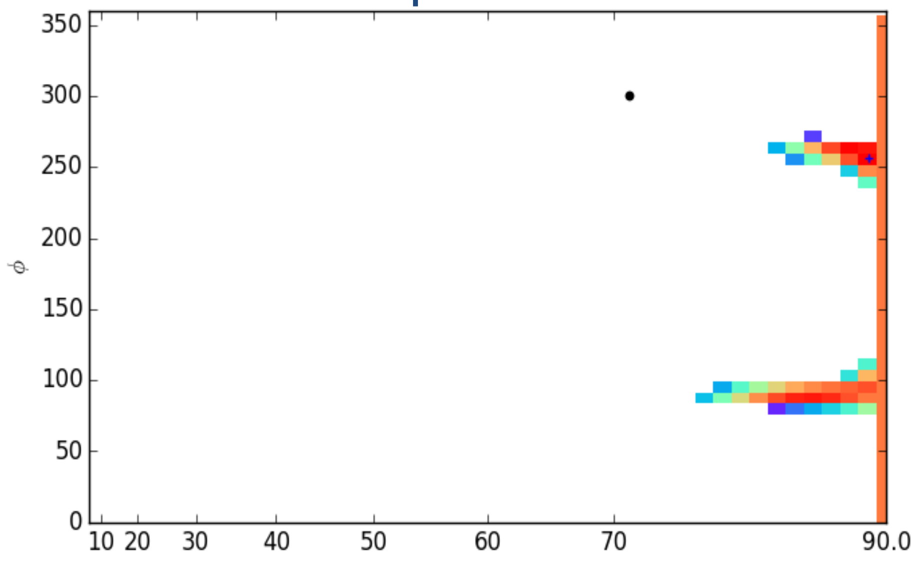




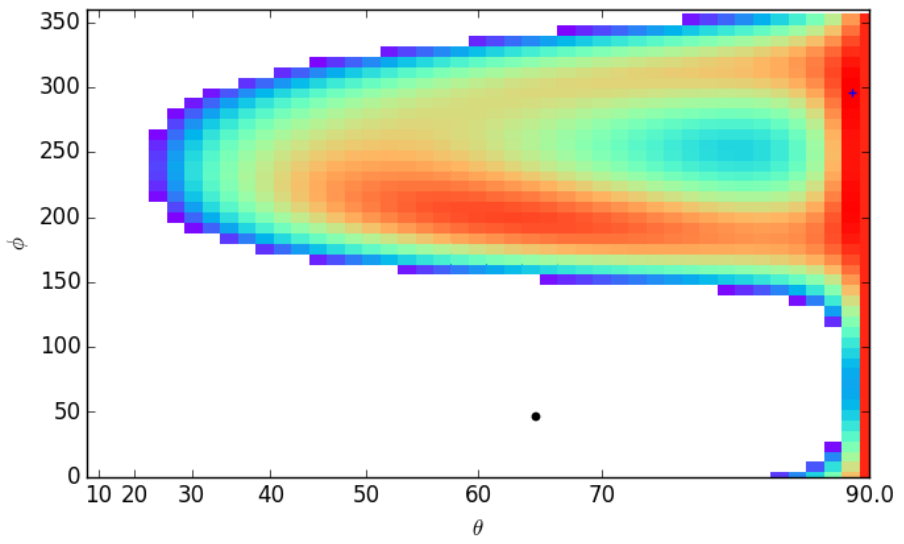


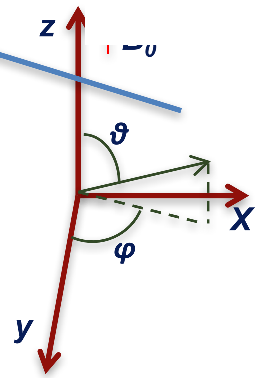
$$\frac{\delta B}{B} \sim 2 \cdot 10^{-5}$$

547 Hz-Pump



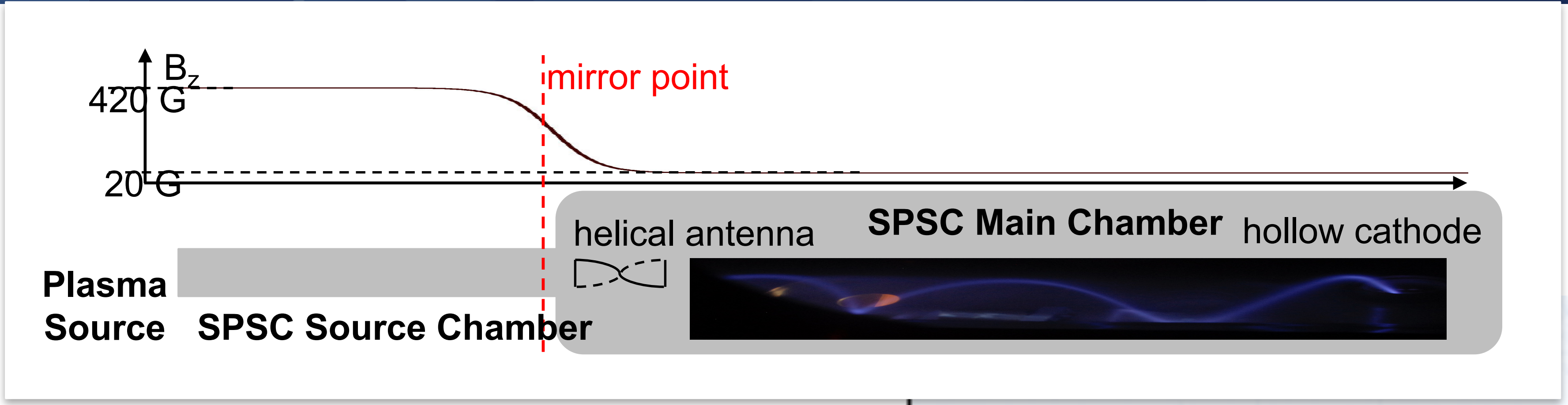
512 Hz-Scattered

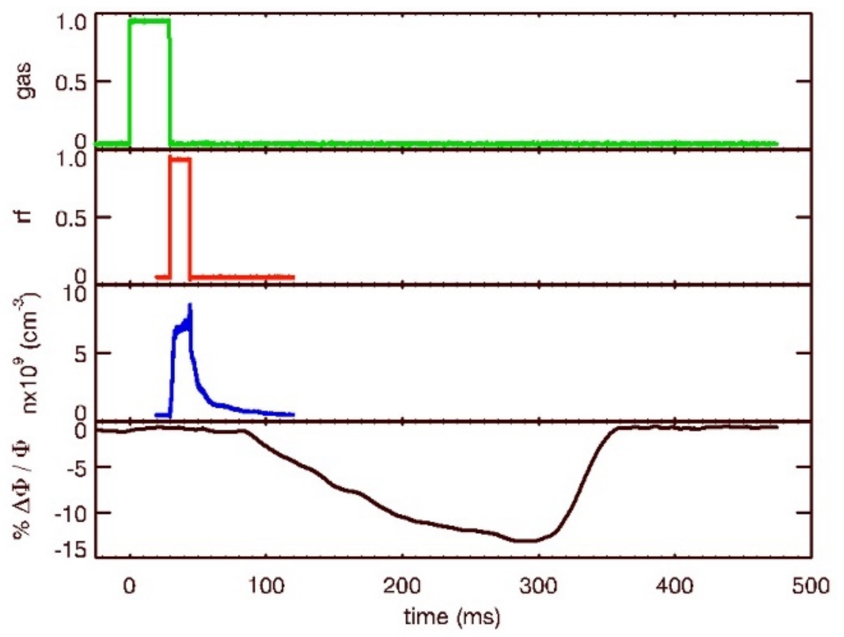


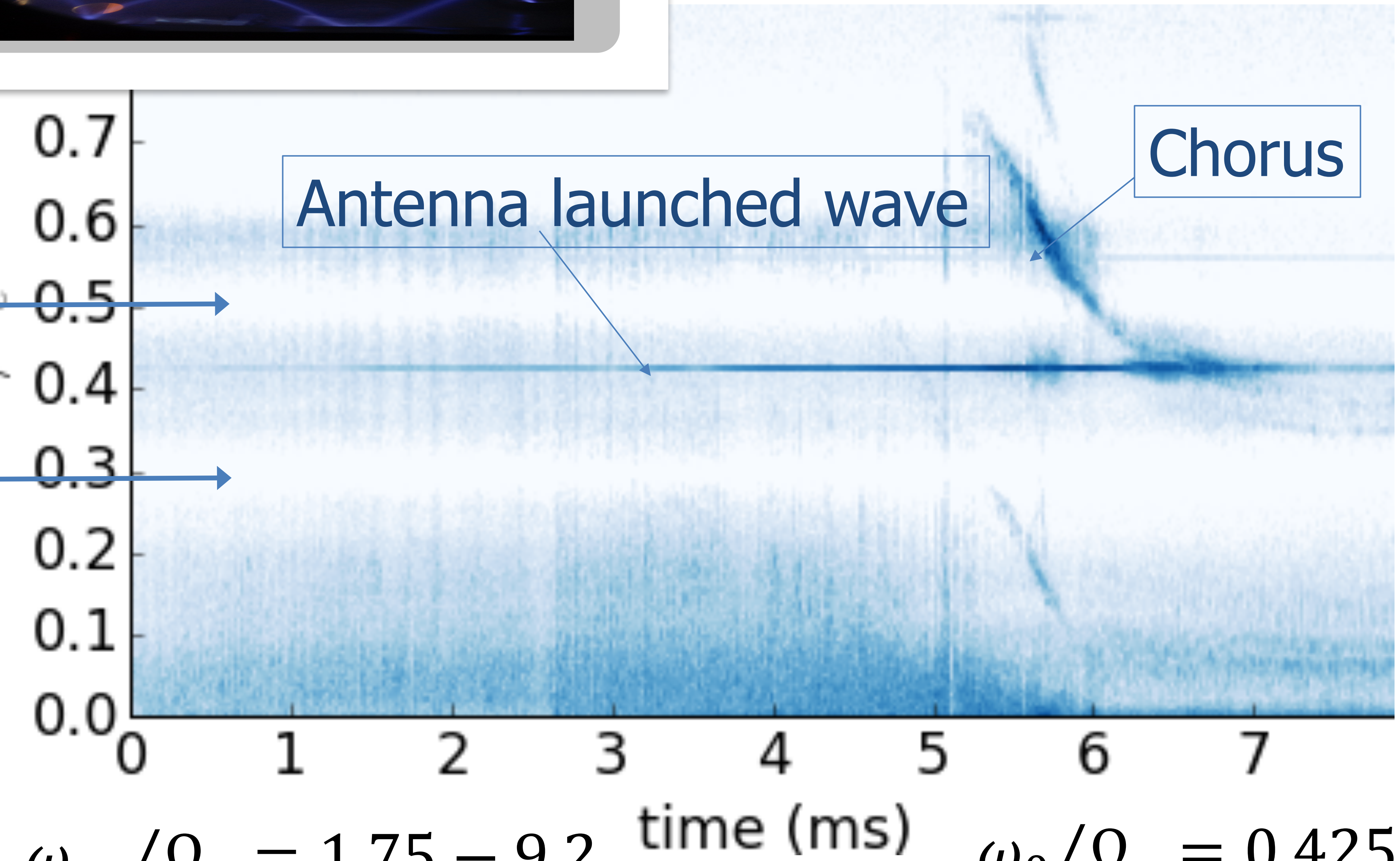


Crabtree *et al.*, *Phys. Plasmas*, **19**, 032903 (2012)
Ganguli *et al.*, *Geophys. Res. Lett.*, **39**, L16105 (2012)

Laboratory Chorus-like whistler emissions: A problem that doesn't fit into the main categories







Nonlinear sub-cyclotron damping

$\alpha = 22°$

$E_b = 1.8\ keV$

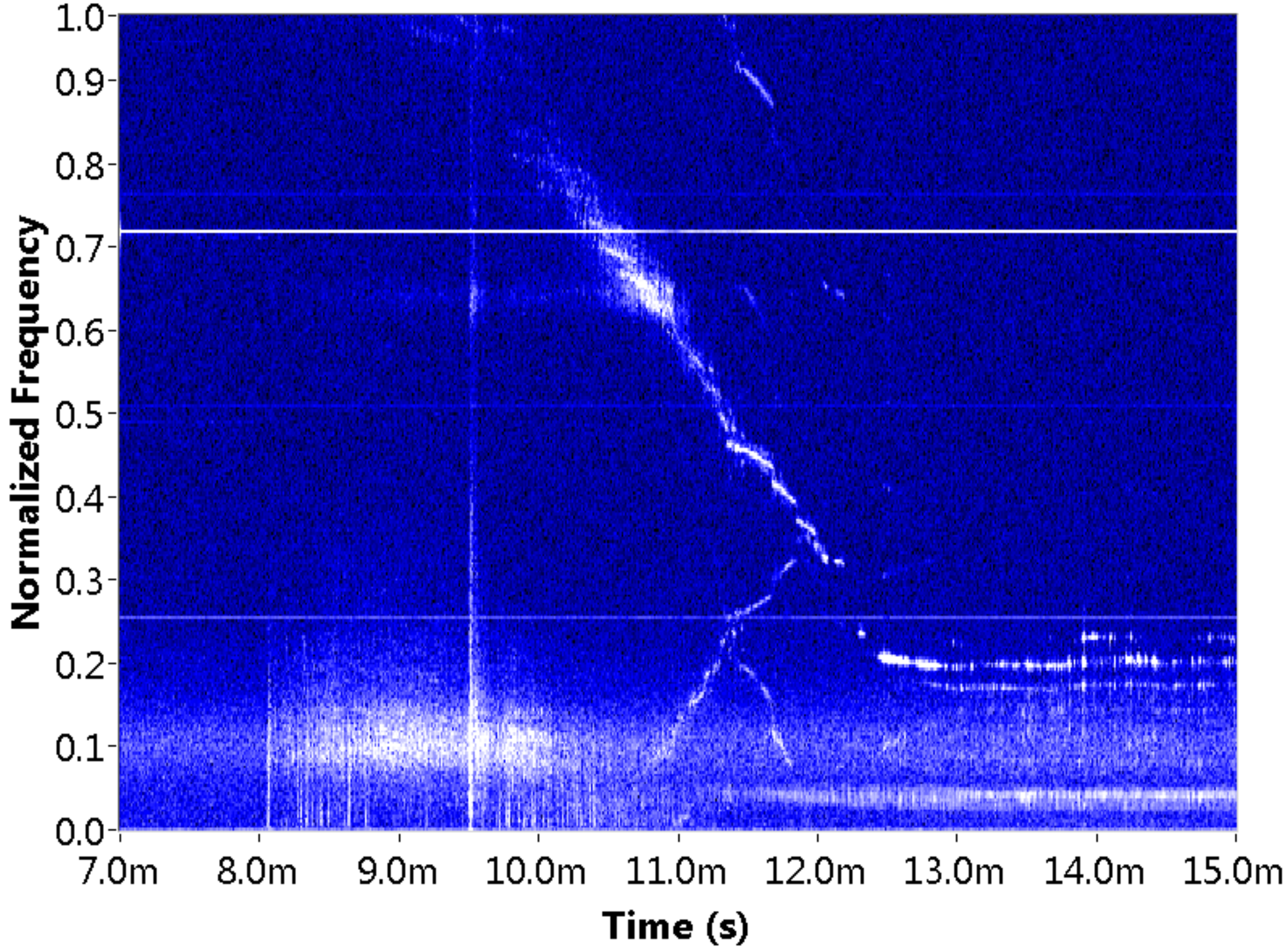
$n_b/n_0 = 0.5 - 13\%$

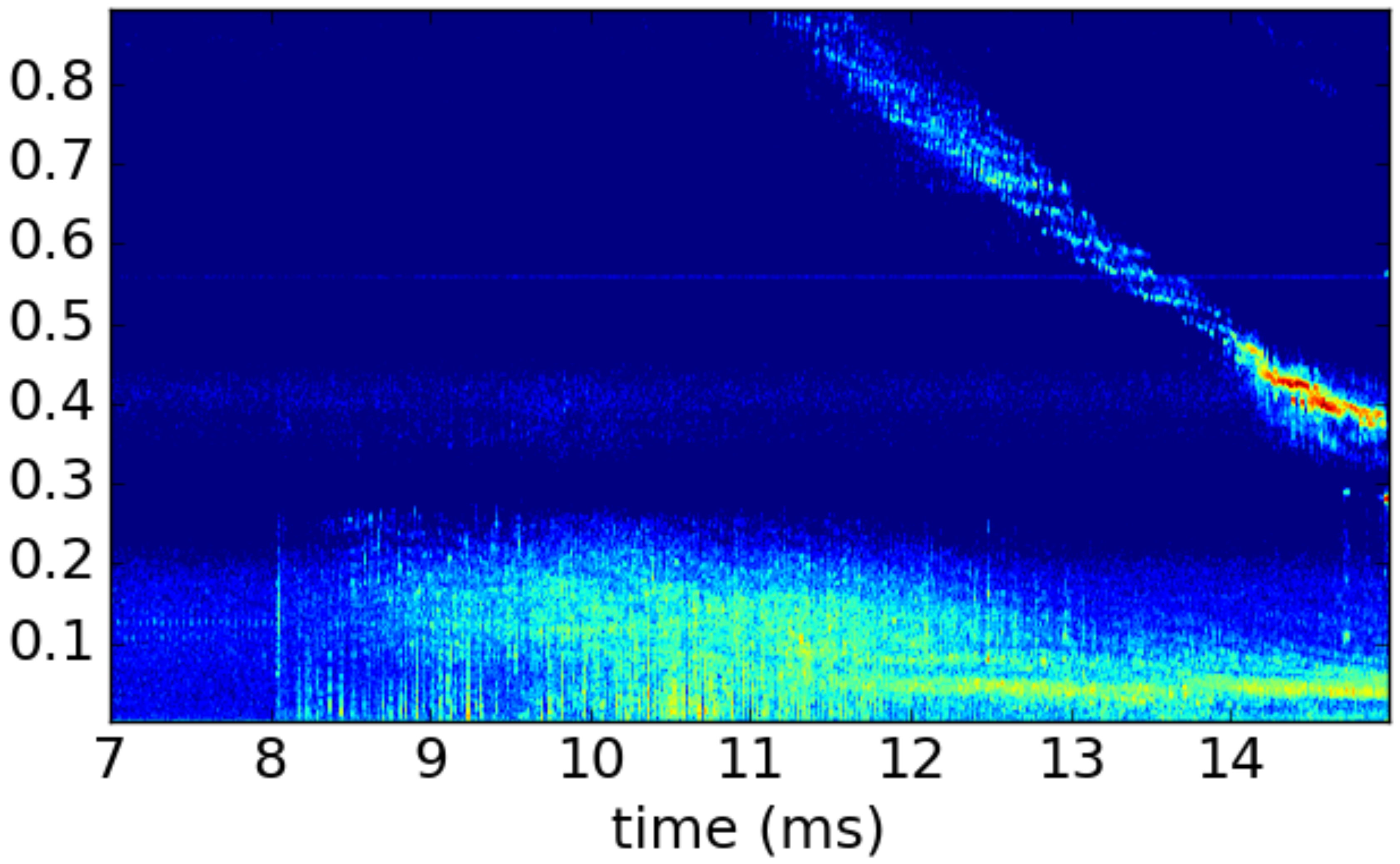
$\omega_{pe}/\Omega_e = 1.75 - 9.2$

$\omega_0/\Omega_e = 0.425$

Pulsed operation means beam density to background density is rising with time

Stair-step Frequency-vs-time Structure in Lab Experiments





Bayesian Analysis of Chorus Element

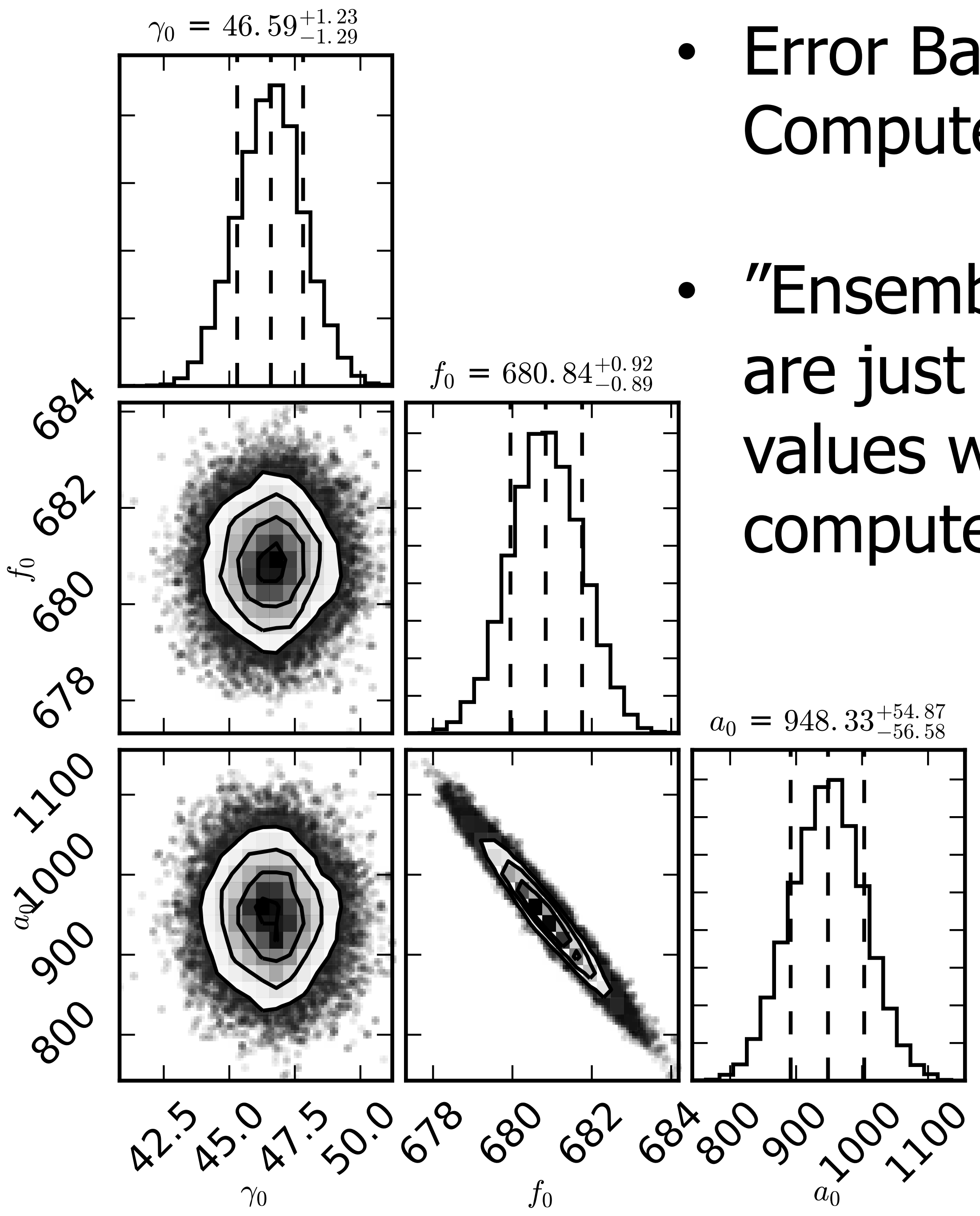
- Compute the probability of model parameters.
- Use model of time series based on perturbative plasma theory,

$$f(t) = \sum_{l=1}^{N_W} e^{\gamma_l t} \left(B_1 \cos(\omega_l t + \alpha_l t^2) + B_2 \sin(\omega_l t + \alpha_l t^2) \right)$$

- Model functions can be made orthogonal over discrete time grid.
- Use all 6 channels of data for parameter determination.
- Integrate probability analytically over amplitude and phase.
- If one assumes model functions

$$f(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$$

- And follows same procedure you recover FFT theory.



- Error Bars Can Be Computed
- "Ensemble Averages" are just expectation values which can be computed

“I will be the greatest jobs president that God ever created.”

Demonstration with Synthetic Data

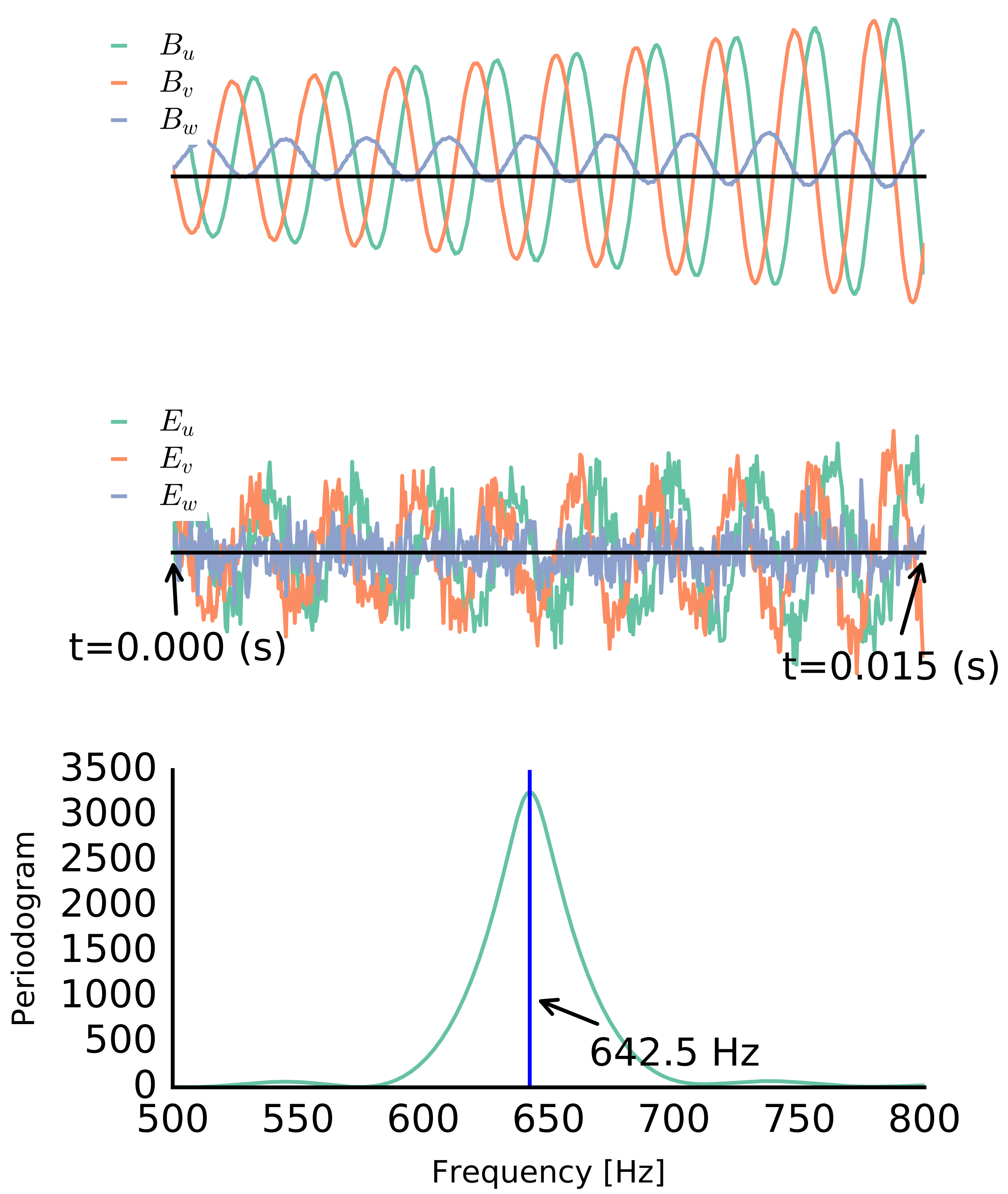
- First wave has frequency 630 Hz, chirp rate 900 Hz/s, growth rate 50 /s, wave-normal angle 11.15°, azimuthal angle 35.3°

- Second wave has frequency 600 Hz, chirp rate 1200 Hz/s, wave normal angle 61.15°, azimuthal angle 65.3°, growth rate -50 /s, with amplitude 10% of first wave at beginning of window

- FFT Bin size is 68.4 Hz

- Gaussian noise is added to each channel

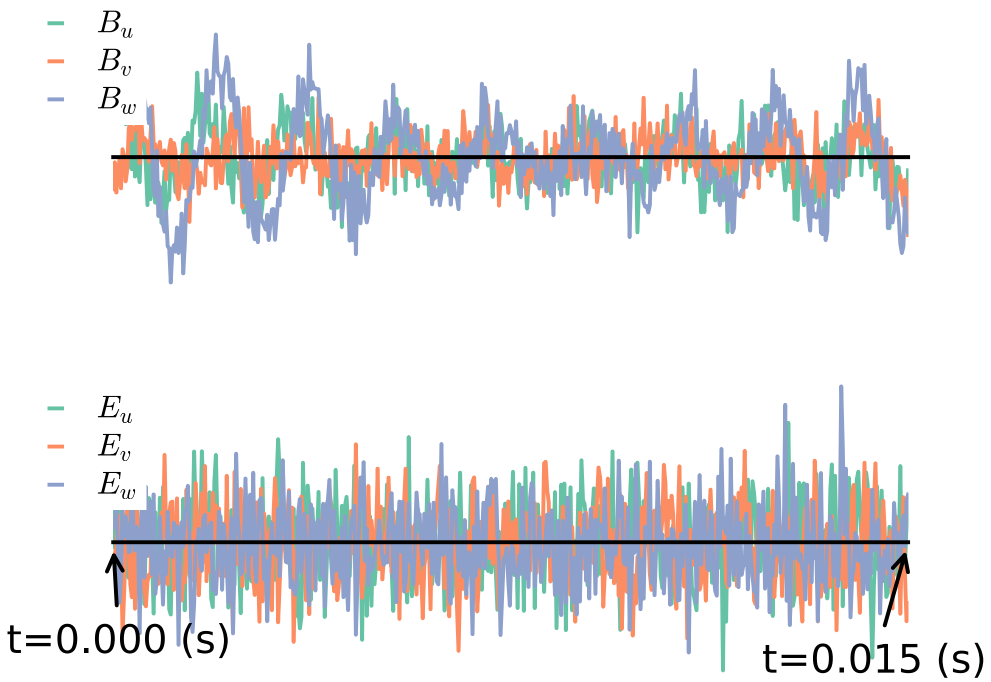
Two Waves

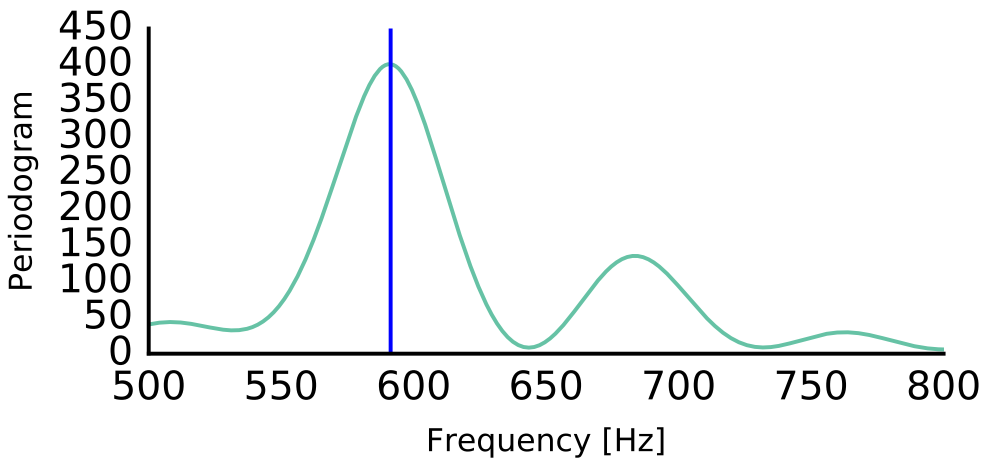


No Reason to suspect more than one signal present

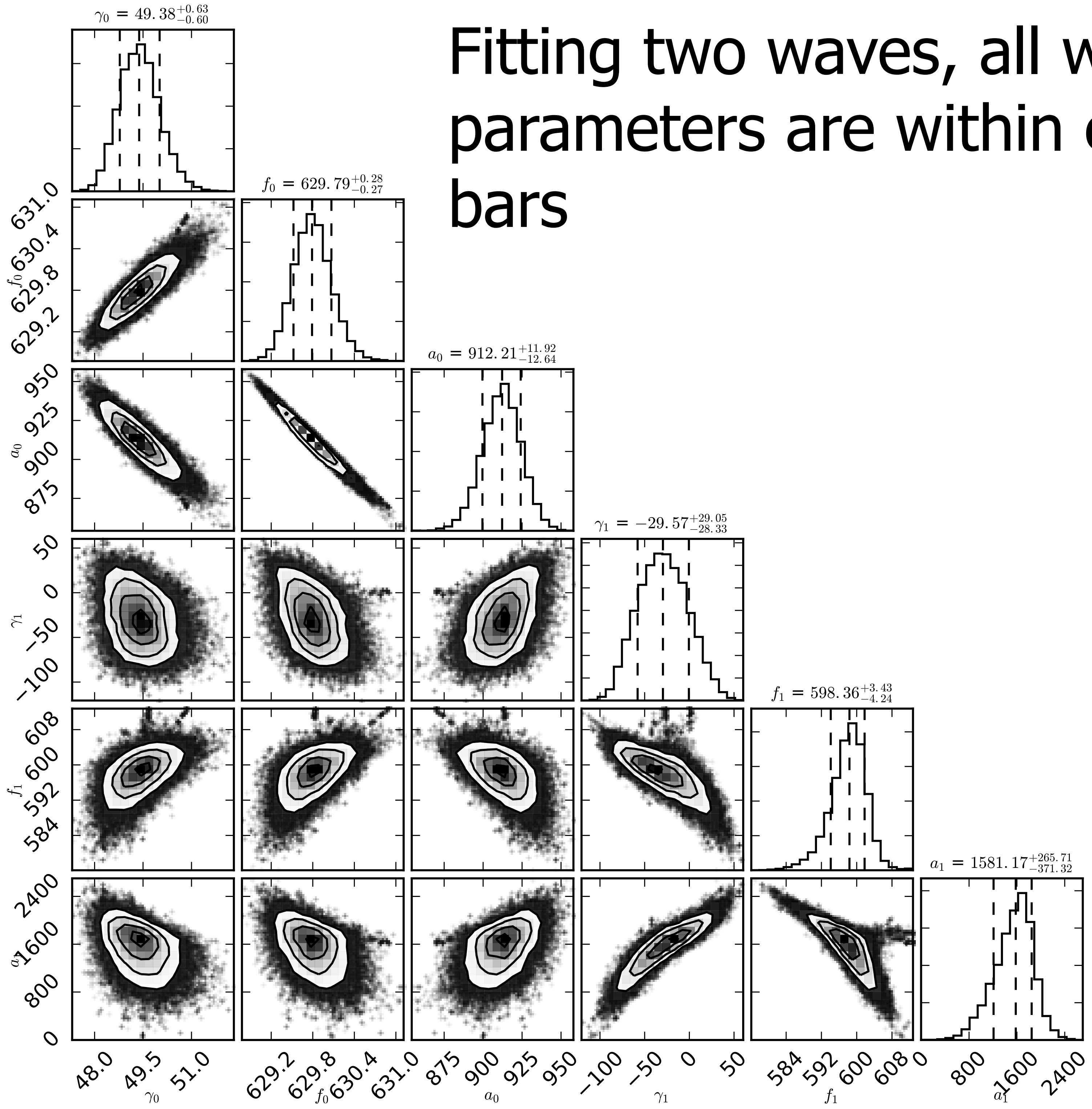
Bayesian Technique Reveals Synthetic Second Wave

Fitting one wave, we can subtract the model from the data, and we can see the smaller signal.

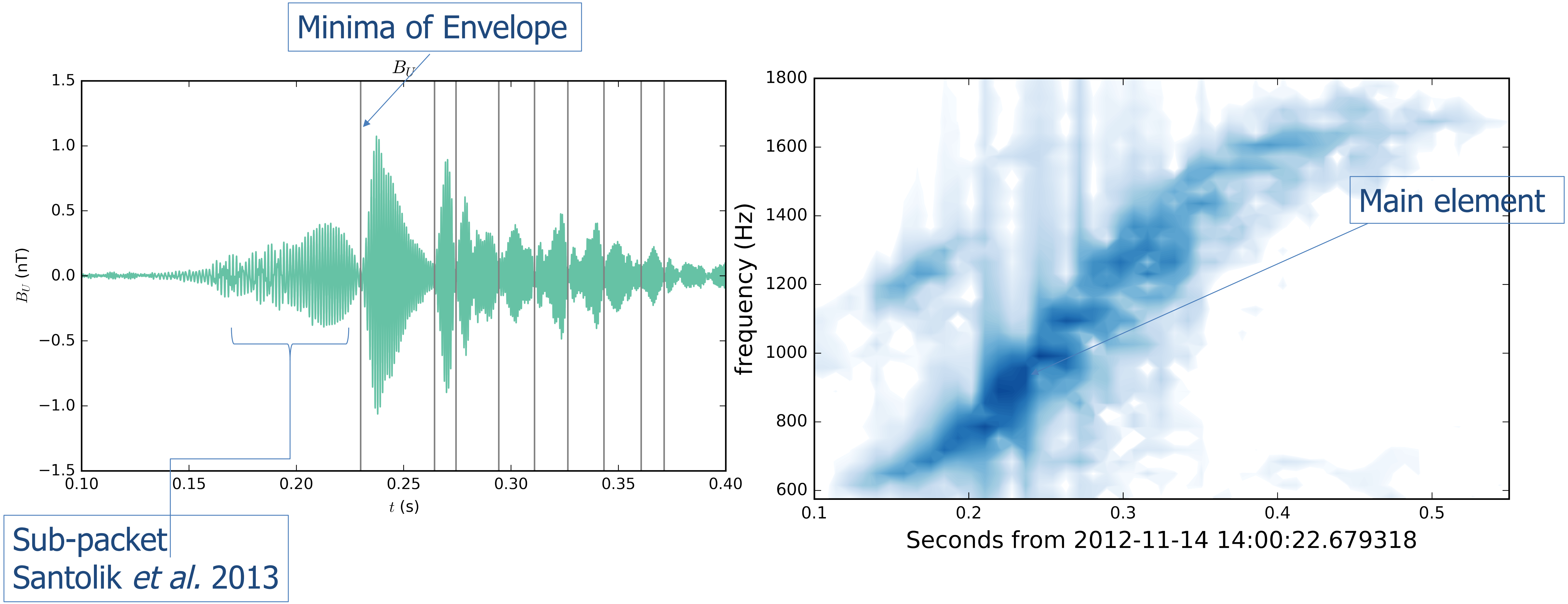




Fitting two waves, all wave-parameters are within error bars



Single chorus element with short time FFT



New Bayesian Spectral Analysis of Chorus Element

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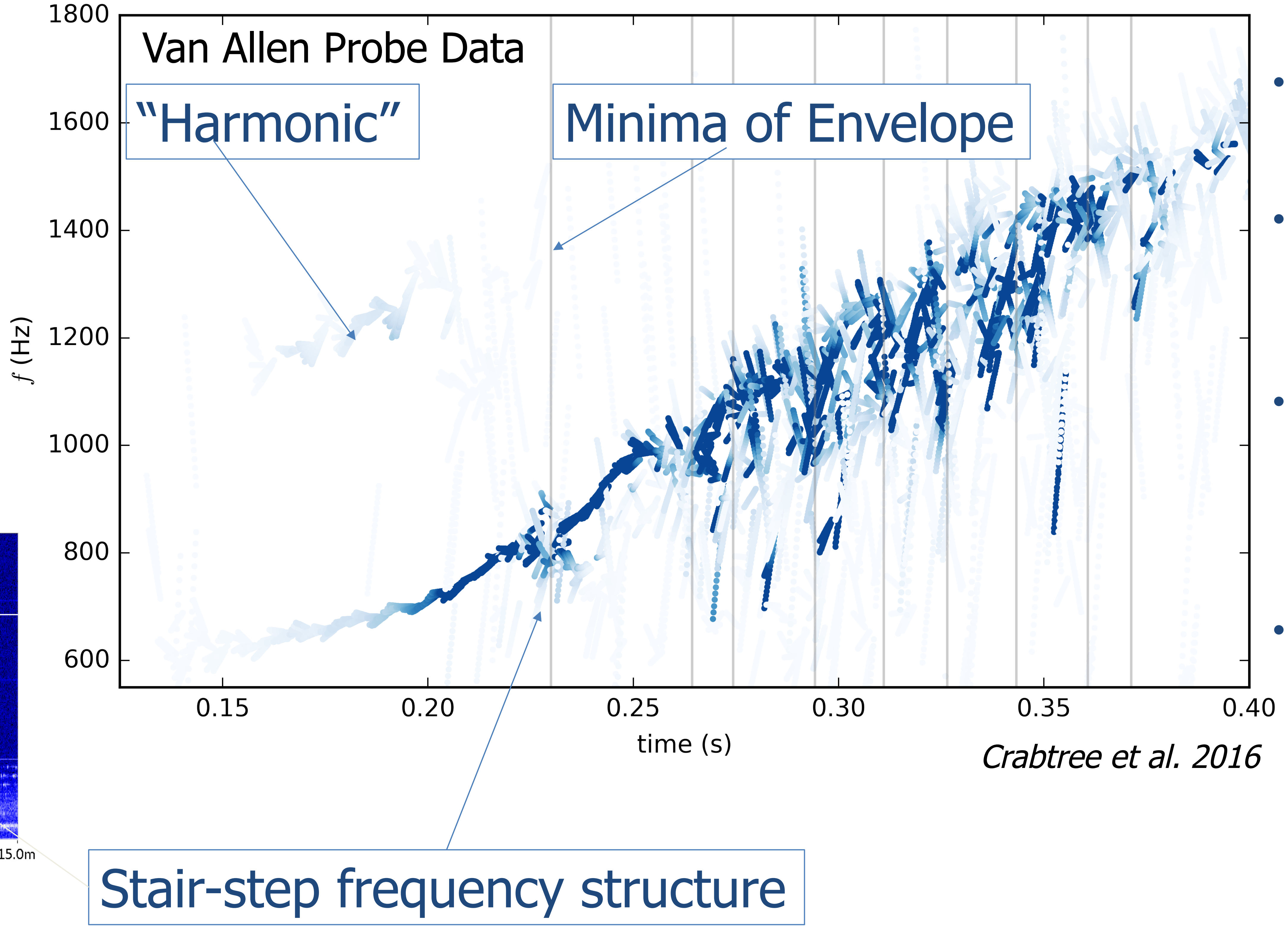
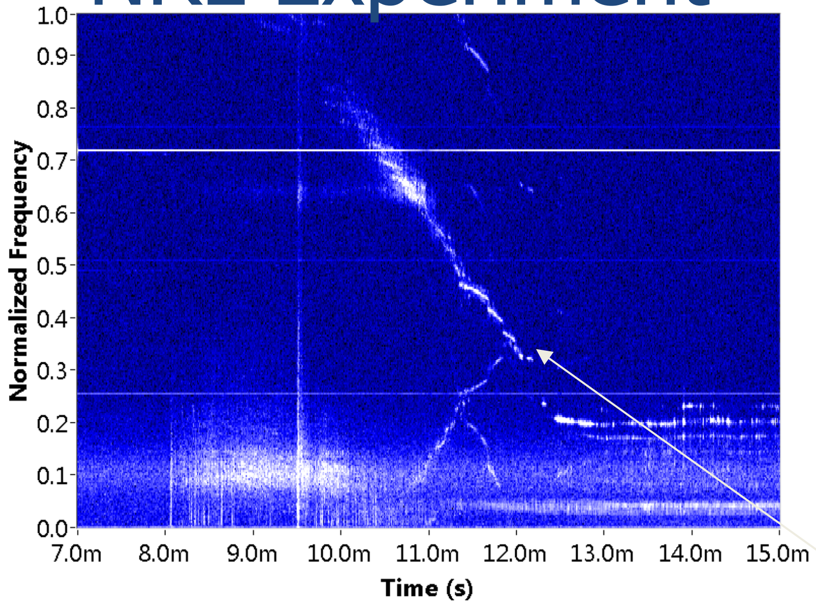


Crabtree et al. 2016

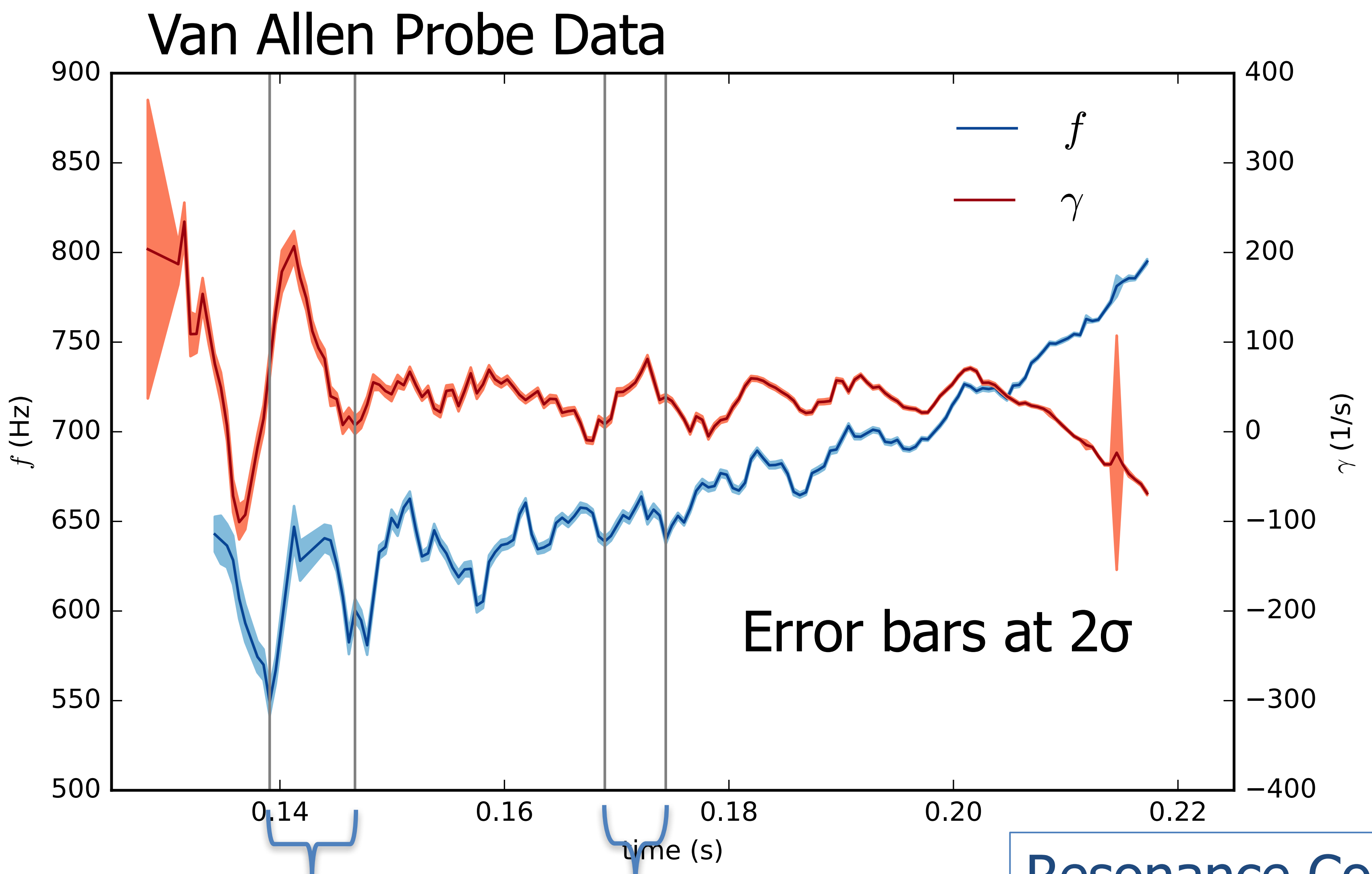
NRL Experiment

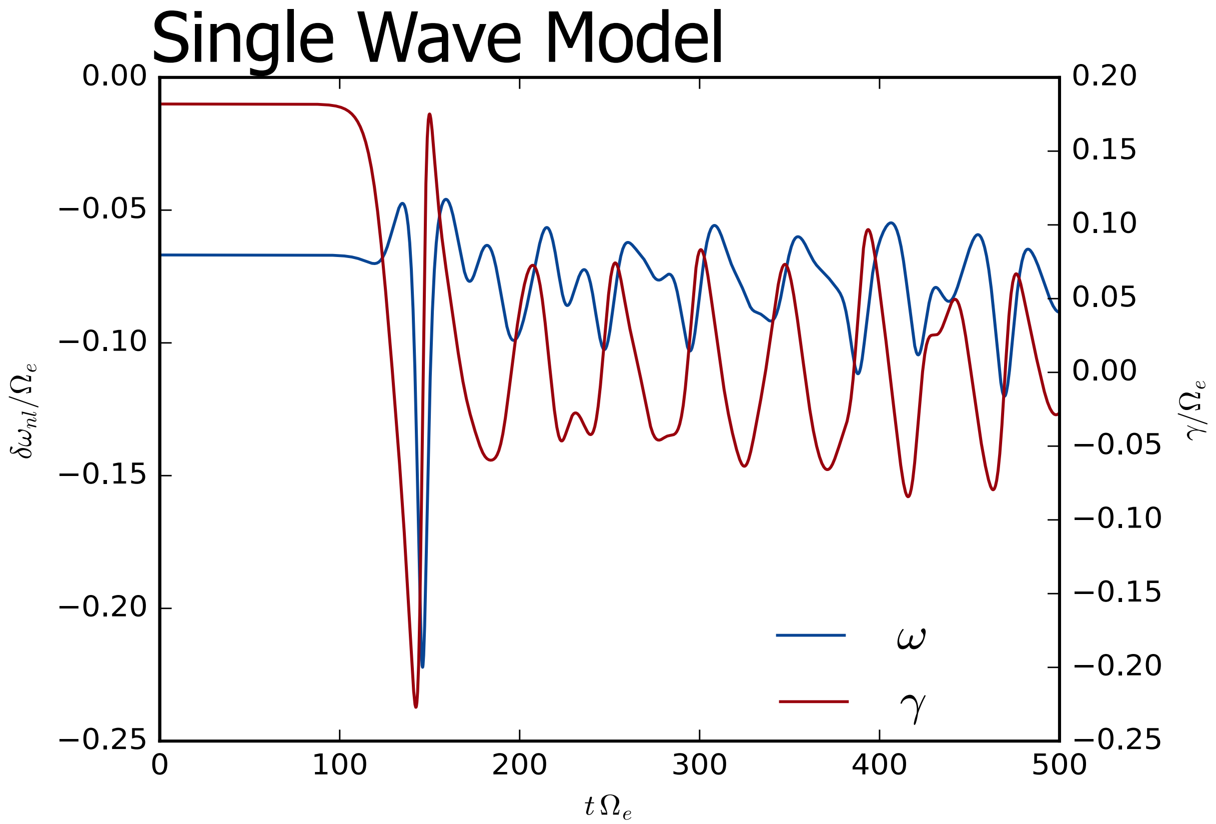


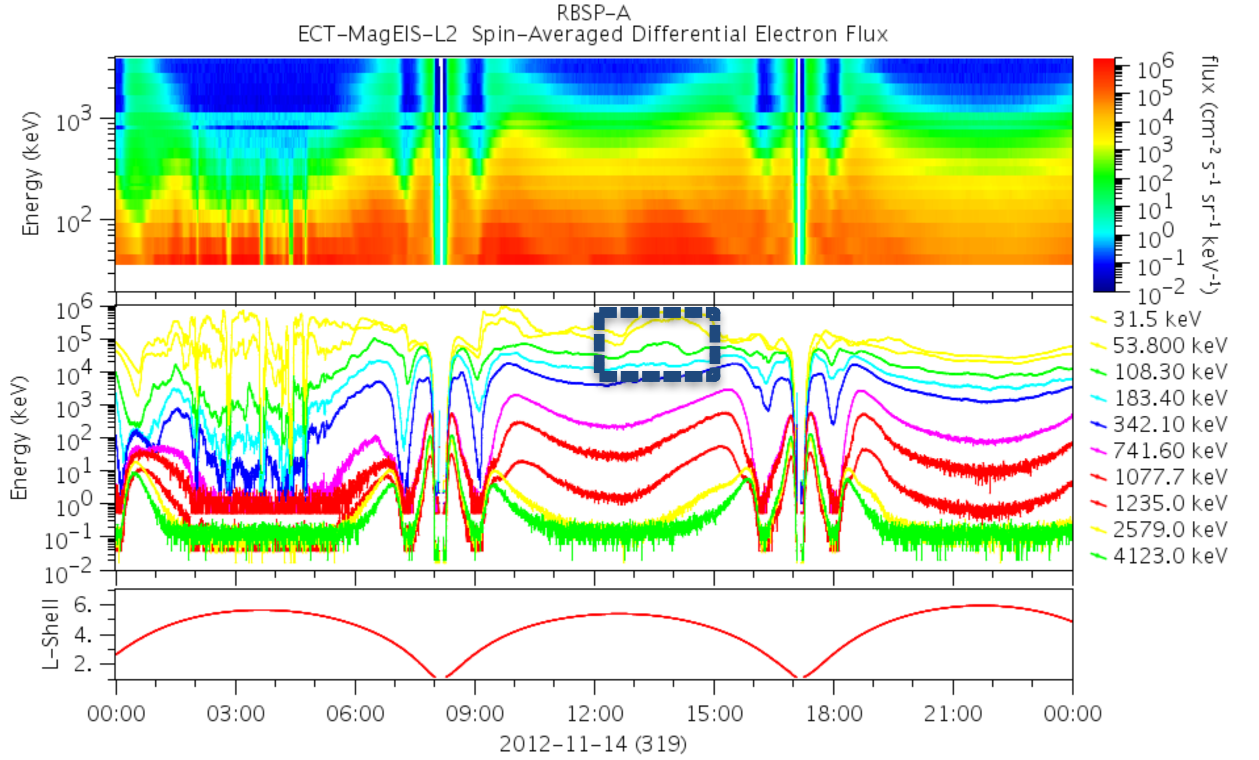
- Intensity corresponds to wave energy density
- Instantaneous frequency is plotted over time window
- Growth rate is used to adjust wave energy density over time window
- Color saturates at 10^{-11} ergs

Trapping Oscillations Observed in Wave Data with Bayesian Method

NRL PPD







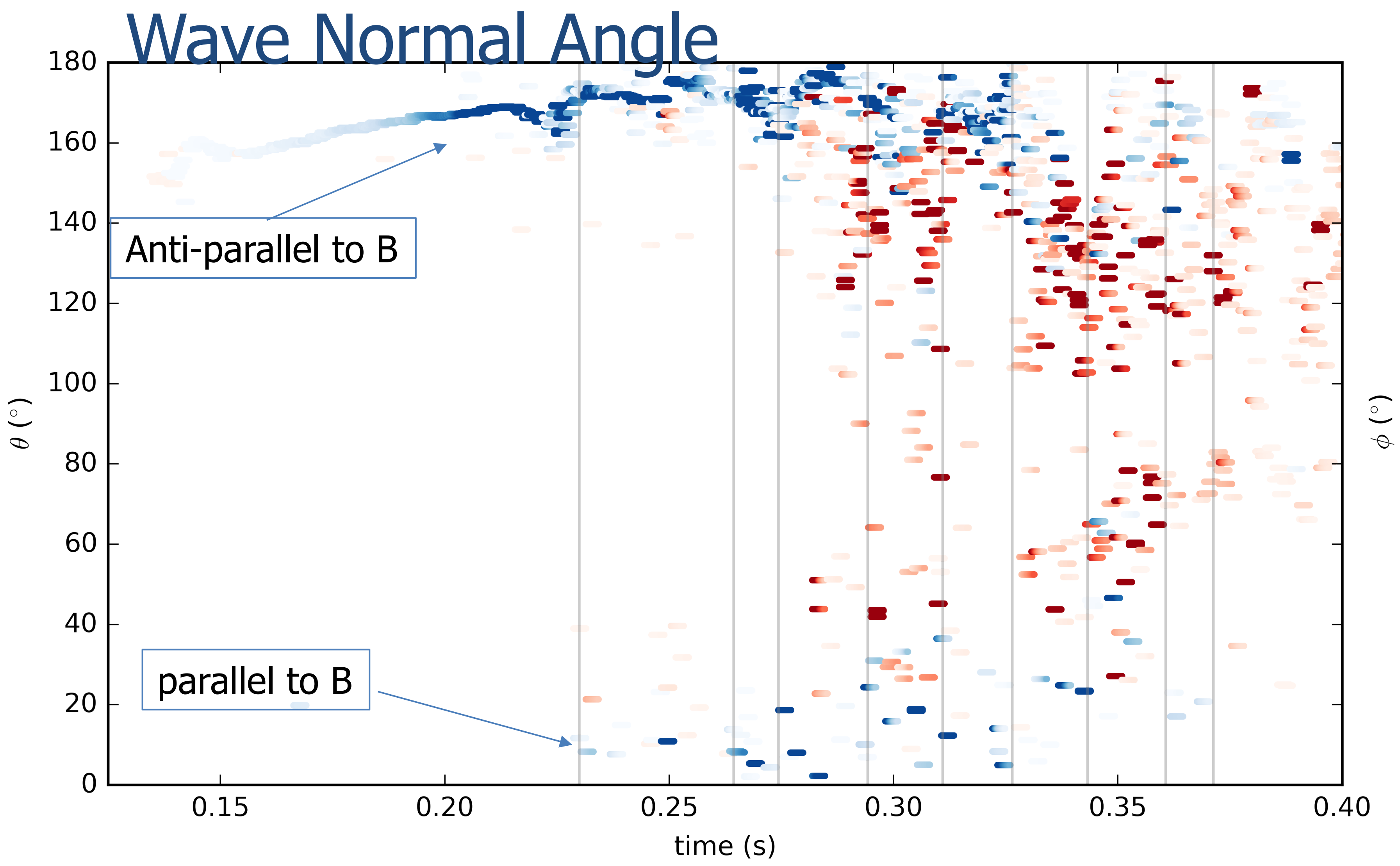
$$\omega_t = \sqrt{k v_\perp \Omega_e B_W / B}$$

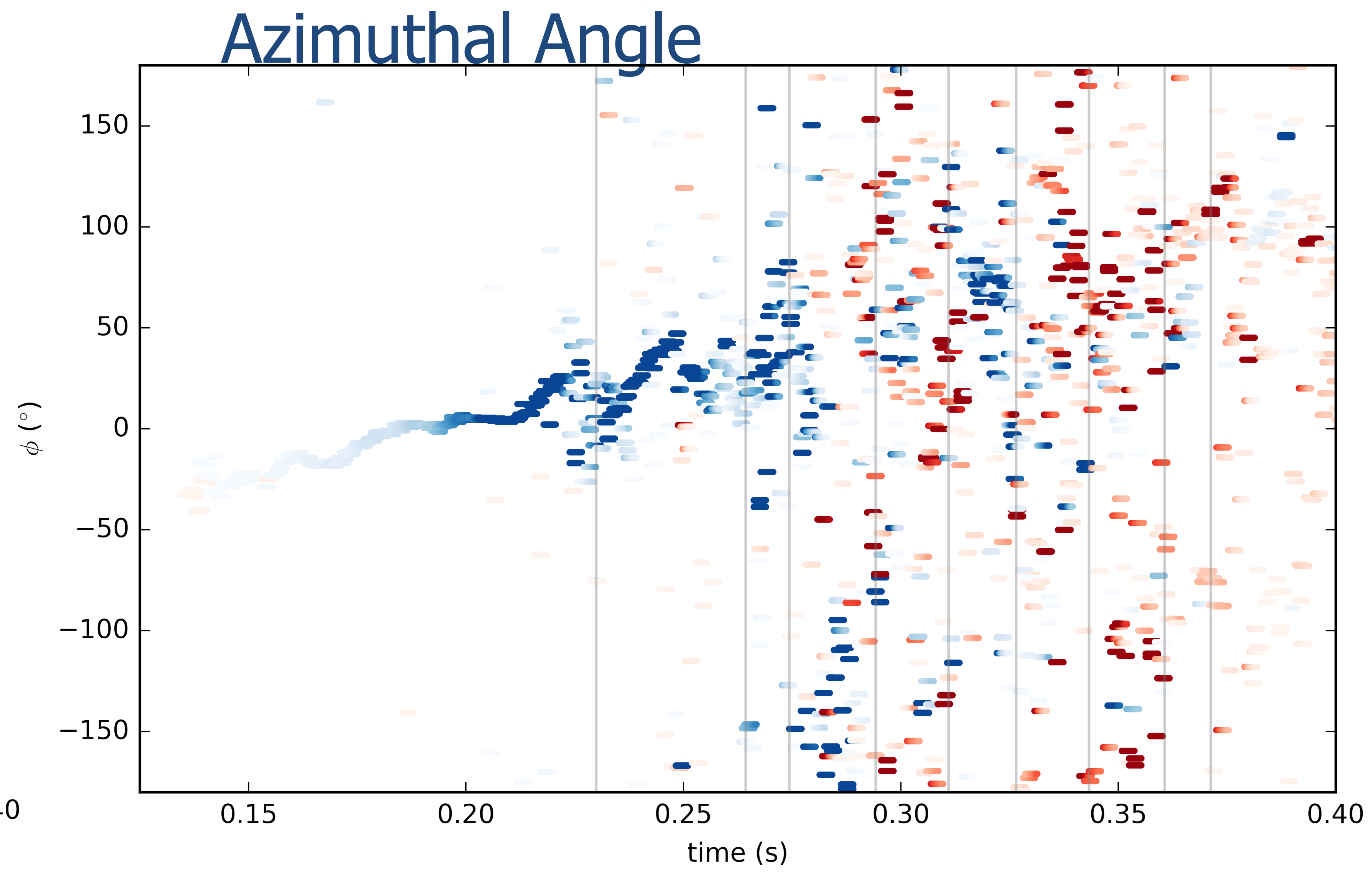
Oscillation's consistent with 10-15 KeV Trapped particles

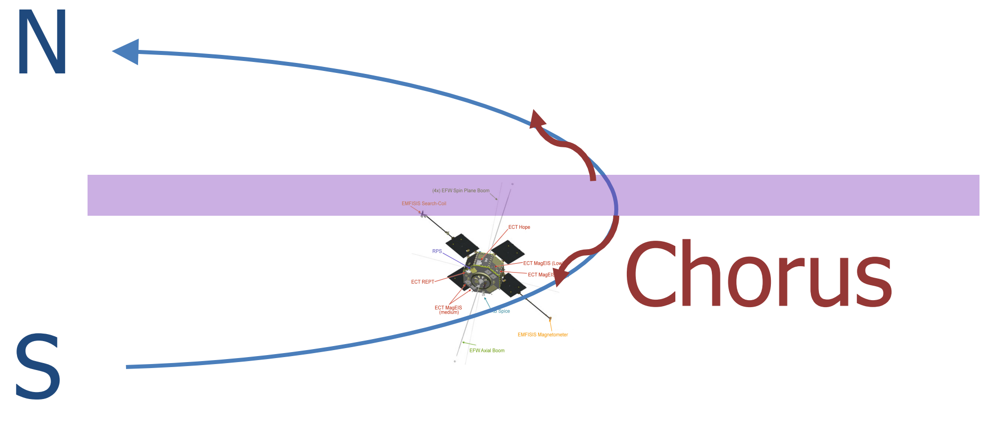
Resonance Condition
$$\omega - k_{||} v_{||} = \Omega_e$$
Gives 50 KeV Energy

Pitch Angle of 25°

Direction of k from Van Allen Probe

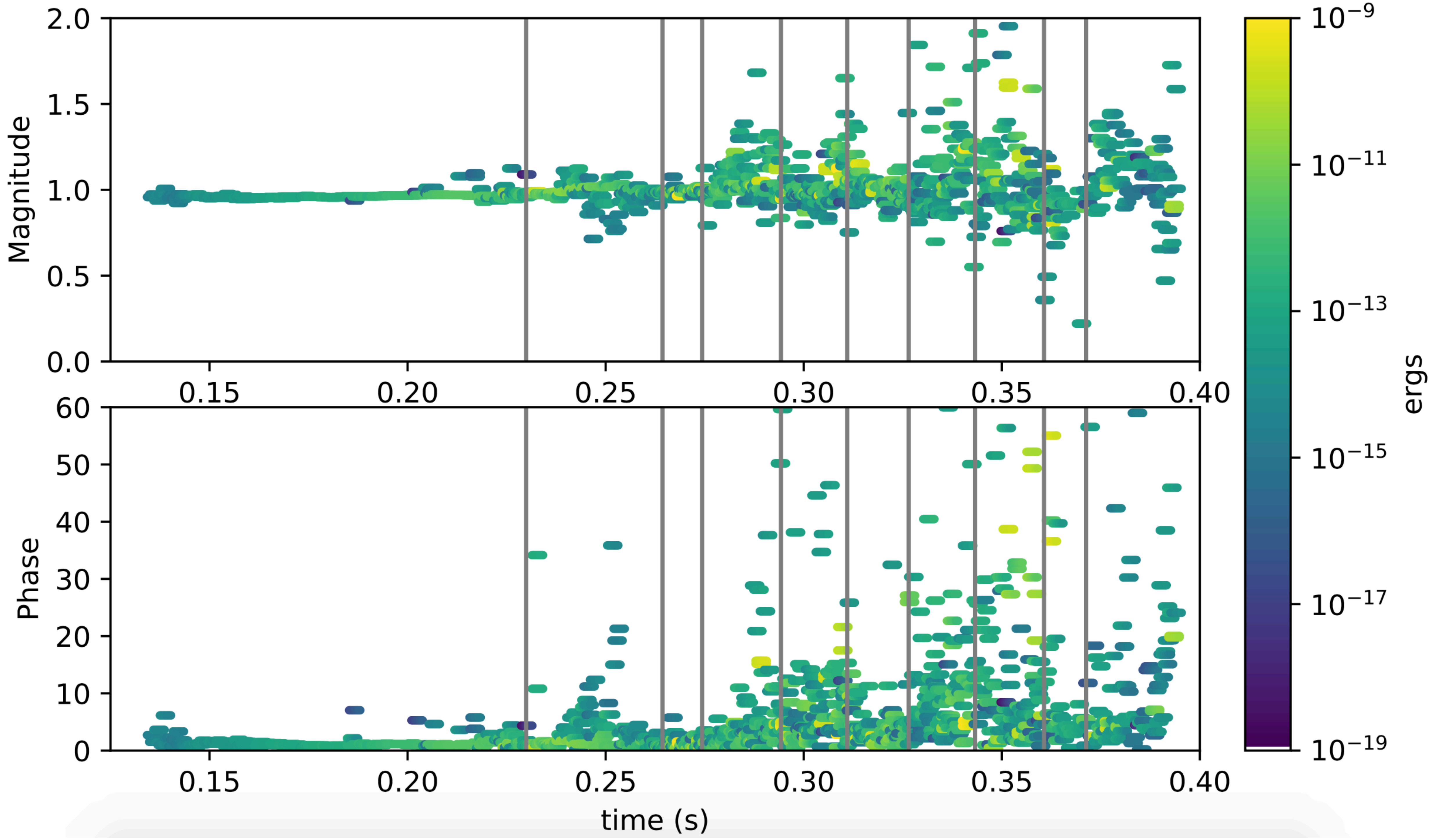






- Color intensity scales with wave energy density.
- Blue agrees with cold plasma linear theory to within 10%
- Red disagrees with cold plasma linear theory by more than 10%
- Angles determined from SVD of magnetic spectral matrix
- Azimuthal angle is in GSM coordinates not space-craft spinning

Comparison with Linear Cold Plasma Theory



Finite Dimensional Hamiltonian Theory for Chorus

NRL PPD

For example, Considerable approximation and simplification leads to a self-consistent wave-particle Hamiltonian:

$$\mathcal{H} = \sum_{i=0}^{M} \Bigg\{ -\Omega_e P_{\phi i} + \frac{p_{zi}^2}{2m}$$

$$- \sum_l \beta_l \sqrt{-2m\Omega_e P_{\phi i} \mathcal{J}_l} \frac{\Omega_e}{B} \left[\rho_{ly} \cos(\Psi_{li} + \varphi_{ly} - \phi) + \rho_{ly} \cos(\Psi_l + \varphi_{ly} + \phi_i)\right.$$

$$\left. -\rho_{lx} \sin(\Psi_{li} + \varphi_{lx} - \phi_i) + \rho_{lx} \sin(\Psi_{li} + \varphi_{lx} + \phi_i)\right]$$

$$- \frac{2\Omega_e}{B} p_{zi} \sum_l \sqrt{\mathcal{J}_l} \hat{\rho}_{lz} \cos(\Psi_{li} + \varphi_{lz})$$

$$+ \frac{m\Omega_e^2}{B^2} \sum_{l,l'} \beta_l \beta_l' \sqrt{\mathcal{J}_l \mathcal{J}_{l'}} \left[\hat{\rho}_{lx}\hat{\rho}_{l'x} \left(\cos(\Psi_{li} - \Psi_{l'i} + \varphi_{lx} - \varphi_{l'x}) + \cos(\Psi_{li} + \Psi_{l'i} + \varphi_{lx} + \varphi_{l'x})\right)\right.$$

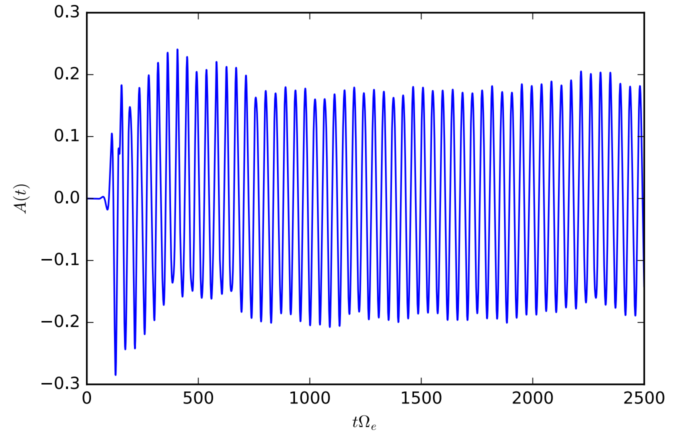
$$+\hat{\rho}_{ly}\hat{\rho}_{l'y} \left(\cos(\Psi_{li} - \Psi_{l'i} + \varphi_{ly} - \varphi_{l'y}) + \cos(\Psi_{li} + \Psi_{l'i} + \varphi_{ly} + \varphi_{l'y})\right)$$

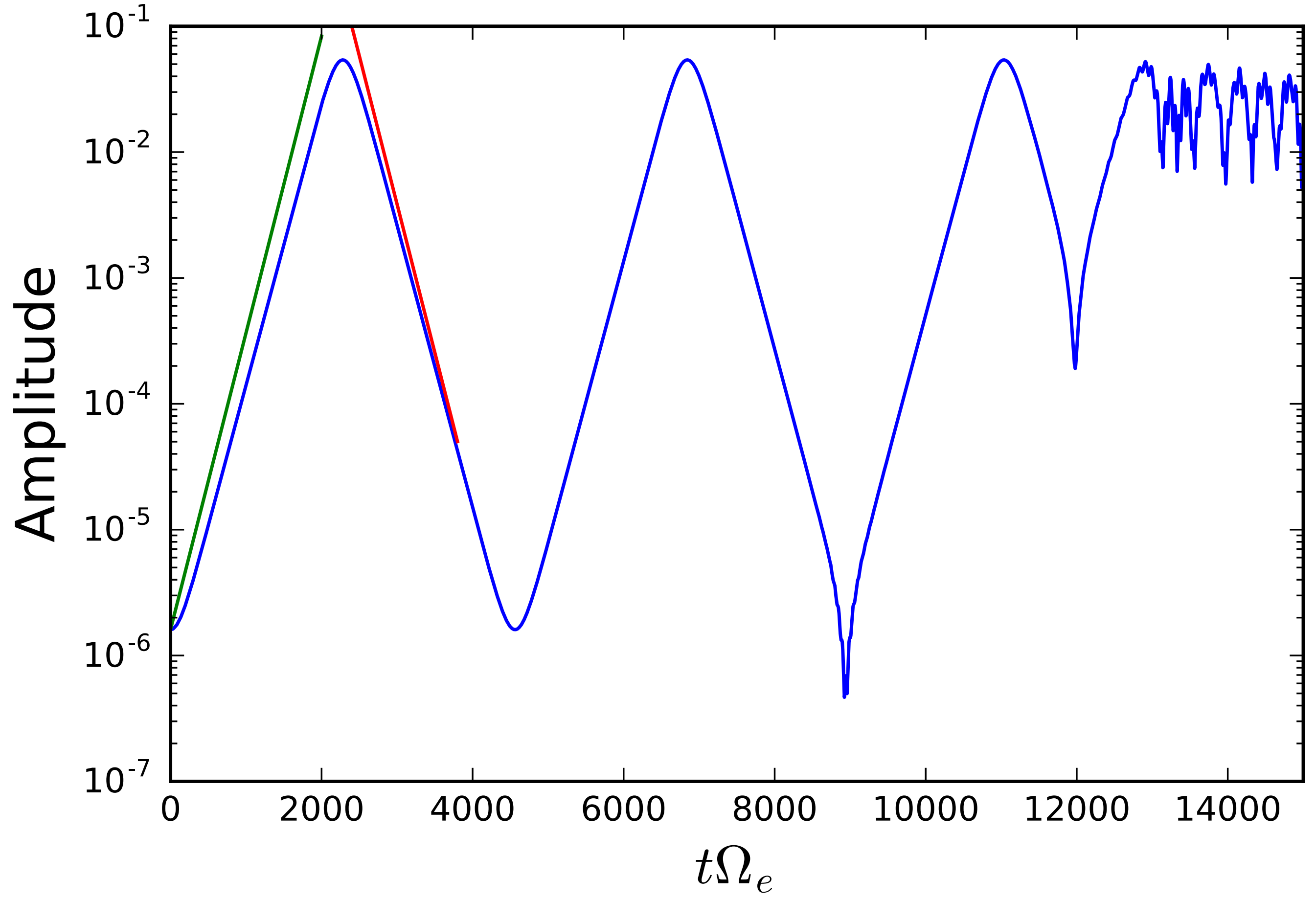
$$\left.+\hat{\rho}_{lz}\hat{\rho}_{l'z} \left(\cos(\Psi_{li} - \Psi_{l'} + \varphi_{lz} - \varphi_{l'z}) + \cos(\Psi_{li} + \Psi_{l'i} + \varphi_{lz} + \varphi_{l'z})\right)\right]\Bigg\} \quad (22)$$

Nonlinear evolution for non-resonant modes

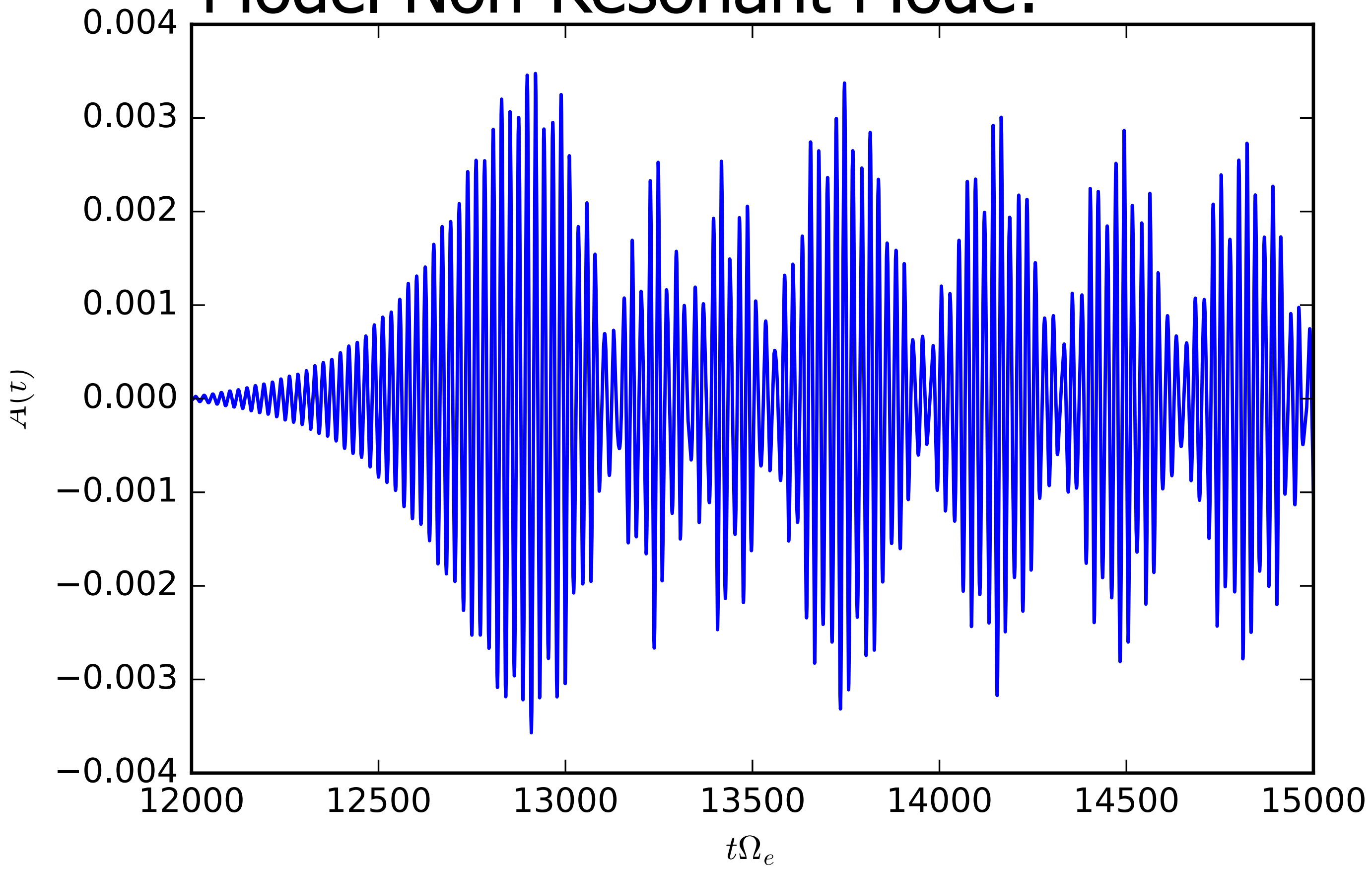
- Pick a spectrally stable fixed point and perturb the beam momentum by 2%
- Calculate linear growth rate from kinetic theory for perturbed beam

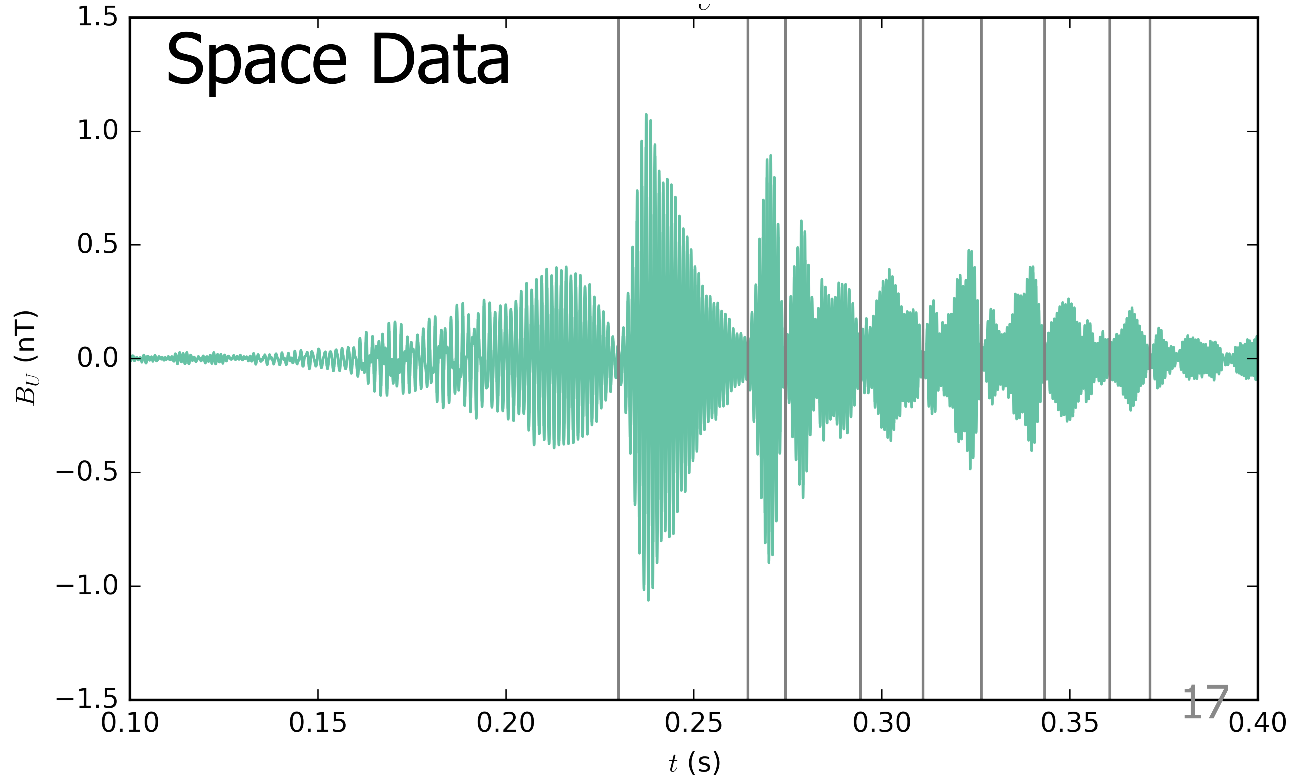
Resonant Mode





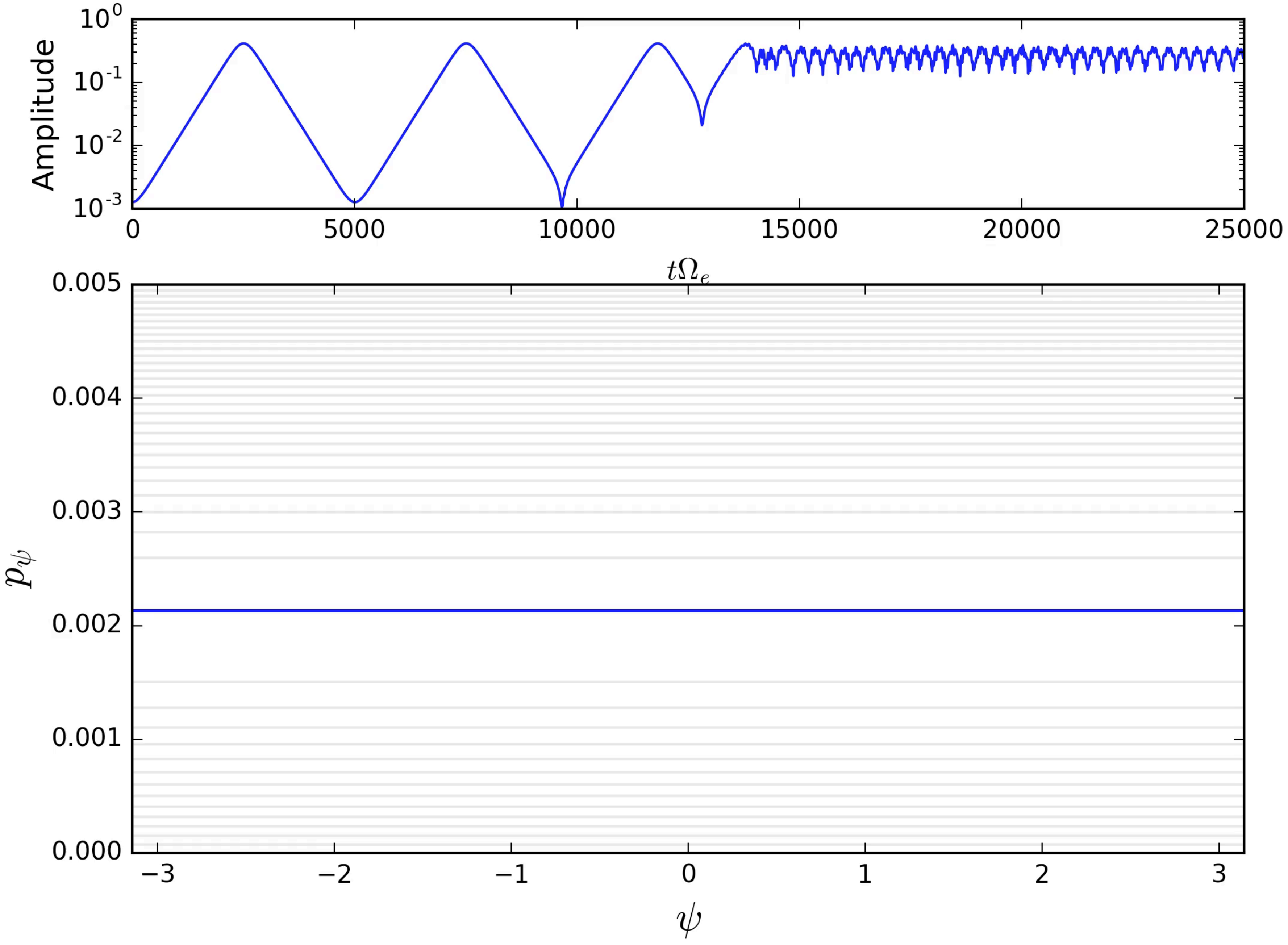
Model Non-Resonant Mode:





Suggests that Wave-Detuning May be source of Sub-packets

Nonlinear evolution of non-resonant instability



- Eigenvectors of linearly growing mode were used to initialize the simulation.
- Linear phase shows particles losing momentum. Inexplicably they gain momentum as wave is damped.
- Nonresonant character of mode can be seen as island slips by.
- At saturation two structures appear, one near fixed point and the other near the seperatrix.

Bayesian perspective formalizes Occam's razor

NRL PPD

Among competing hypotheses, the one with the fewest assumptions should be selected ***that fits the data the best.***

Suppose you have models $\{f_1, f_2, \ldots f_s\}$ for the data D. Then,

$$\sum_{i=1}^{s} P(f_i|D,I) = 1$$

thus one can calculate which model has best probability of describing the data. But how to calculate probability?

$$P(f_i|D,I) = \frac{P(f_i|I)P(D|f_j,I)}{P(D|I)} \qquad P(D|I) = \sum_{i=1}^{s} P(f_j|I)P(D|f_j,I)$$

And where each model f_i has some set of parameters ω

$$P(D|f_j,I) = \int d\omega P(\omega|I)P(D|\omega,I)$$

William of Ockham
(1287-1347)

Conclusions/Comments/Questions

- We have a long tradition of theory leading lab experiment and lab experiment leading space observation.
  - But that worked best for linear theory? Perhaps experiment and space observation can lead theory now?

- Bayesian Techniques offer a way to extract a lot of information from a data signal given a physical theory (need not be expressible in closed form)
  - Standard tools available now to perform numerical computations of probabilities

- Bayesian Techniques offer a way to compare alternative theories using a framework that implies Occam's razor